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The effects of hygrothermal conditions on the postbuckling of shear deformable laminated cylindrical shells

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Abstract

The effect of hygrothermal conditions on the buckling and postbuckling of shear deformable laminated cylindrical shells subjected to combined loading of axial compression and external pressure is investigated using a micro-to-macro-mechanical analytical model. The material properties of the composite are affected by the variation of temperature and moisture, and are based on a micro-mechanical model of a laminate. The governing equations are based on Reddy's higher order shear deformation shell theory with von Kármán–Donnell-type of kinematic nonlinearity and including hygrothermal effects. The nonlinear prebuckling deformations and initial geometric imperfections of the shell are both taken into account. A boundary layer theory of shell buckling is extended to the case of shear deformable laminated cylindrical shells under hygrothermal environments and a singular perturbation technique is employed to determine the interactive buckling loads and postbuckling equilibrium paths. The numerical illustrations concern the postbuckling behaviour of perfect and imperfect, moderately thick, cross-ply laminated cylindrical shells under different sets of environmental conditions. The results show that the hygrothermal environment has a significant effect on the interactive buckling load as well as postbuckling response of the shell. In contrast, it has a small effect on the imperfection sensitivity of the shell with a very small geometric imperfection. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Hygrothermal effect; Postbuckling; Moderately thick laminated cylindrical shell; Boundary layer theory of shell buckling; Singular perturbation technique

1. Introduction

In recent years, fibre-reinforced composite laminated shell structures have been widely used in the aerospace, marine, automobile and other engineering industries. During the operational life, the variation of temperature and moisture reduces the elastic moduli and degrades the strength of the laminated material. As a result, a careful evaluation of the effects of environmental exposure is required to find the nature and extent of their deleterious effects upon performance.

Many postbuckling studies, based on classical shell theory, of composite laminated thin cylindrical shells subjected to mechanical or thermal loading or their combinations are available in the literature, see, for

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example Birman and Bert (1993), and Shen (1997a,b,c, 1998, 1999). Relatively few studies involving the application of shear deformation shell theory to postbuckling analysis can be found in Iu and Chia (1988), Reddy and Savoia (1992), Eslami et al. (1998) and Eslami and Shariyat (1999). In these studies the material properties are considered to be independent of temperature. However, studies of temperature and moisture effects on the buckling loads of laminated flat and cylindrical panels are limited in number (see, e.g., Whitney and Ashton, 1971; Flaggs and Vinson, 1978; Snead and Palazotto, 1983; Lee and Yen, 1989; Ram and Sinha, 1992; Chao and Shyu, 1996), and all these studies assumed perfectly initial configurations. In fact, many shells are subjected to high load levels that may result in nonlinear load-deflection relationships, even if the shell has a moderate thickness. These shells may also have unavoidable initial imperfections. Shen (2000) gave a full nonlinear postbuckling analysis of composite laminated cylindrical shells subjected to combined loading of axial compression and external pressure under hygrothermal conditions. It should be noted that in the above study the shell is considered as being relatively thin and therefore the transverse shear deformation is usually not accounted for. The present study extends the previous work to the case of moderately thick laminated cylindrical shells under combined loads and under hygrothermal conditions.

In the present study, both ambient temperature and moisture are assumed to have a uniform distribution. The shell is fully saturated such that the variation of temperature and moisture are independent of time and position. The material properties are assumed to be functions of temperature and moisture. The thermal expansion and swelling coefficients are based on a micro-mechanical model of a laminate (Tsai and Hahn, 1980). In terms of a macro-mechanical model, the governing equations are based on Reddy's higher order shear deformation shell theory with von Kármán–Donnell-type of kinematic nonlinearity and including hygrothermal effects. A boundary layer theory of shell buckling, suggested by Shen and Chen (1988, 1990), is extended to the case of shear deformable laminated cylindrical shells under hygrothermal environments and a singular perturbation technique is employed to determine the interactive buckling loads and postbuckling equilibrium paths. The nonlinear prebuckling deformations and initial geometric imperfections of the shell are both taken into account but, for simplicity, the form of initial geometric imperfection is assumed to be the same as the initial buckling mode of the shell.

2. Theoretical development

Consider a cylindrical shell with mean radius R, length L and thickness t, which consists of N plies, exposed to moisture and elevated temperature, and is subjected to two mechanical loads, an axial load P_0 and a uniform external pressure q. The shell is referred to a coordinate system (X, Y, Z), in which X and Y are in the axial and circumferential directions of the shell and Z is in the direction of the inward normal to the middle surface, the corresponding displacement designated by $\bar{U}$, $\bar{V}$ and $\bar{W}$. $\bar{\Psi}_x$ and $\bar{\Psi}_y$ are the rotations of normals to the middle surface with respect to the Y- and X-axes, respectively. The origin of the coordinate system is located at the end of the shell on the middle plane. The shell is assumed to be relatively thick, geometrically imperfect. Denoting the initial geometric imperfection by $\bar{W}^*(X, Y)$, let $\bar{W}(X, Y)$ be the additional deflection and $\bar{F}(X, Y)$ be the stress function for the stress resultants defined by $\bar{N}_x = \bar{F}_{,yy}$, $\bar{N}_y = \bar{F}_{,xx}$ and $\bar{N}_{xy} = -\bar{F}_{,xy}$, where a comma denotes partial differentiation with respect to the corresponding coordinates.

Reddy and Liu (1985) developed a simple higher order shear deformation shell theory, in which the transverse shear strains are assumed to be parabolically distributed across the shell thickness and which contains the same dependent unknowns as in the first order shear deformation theory. Based on Reddy's higher order shear deformation theory with von Kármán–Donnell-type kinematic relations and including hygrothermal effects, governing differential equations are derived and can be expressed in terms of a stress function $\bar{F}$, two rotations $\bar{\Psi}_x$ and $\bar{\Psi}_y$, and transverse displacement $\bar{W}$, along with initial geometric imperfection $\bar{W}^*$. For moderately thick cross-ply laminated cylindrical shells, they are

$$\tilde{L}_{11}(\bar{W}) - \tilde{L}_{12}(\bar{\Psi}_x) - \tilde{L}_{13}(\bar{\Psi}_y) + \tilde{L}_{14}(\bar{F}) - \tilde{L}_{15}(\bar{N}^{\mathrm{H}}) - \tilde{L}_{16}(\bar{M}^{\mathrm{H}}) - \frac{1}{R}\bar{F}_{,xx} = \tilde{L}(\bar{W} + \bar{W}^*, \bar{F}) + q \tag{1}$$

$$\tilde{L}_{21}(\bar{F}) + \tilde{L}_{22}(\bar{\Psi}_x) + \tilde{L}_{23}(\bar{\Psi}_y) - \tilde{L}_{24}(\bar{W}) - \tilde{L}_{25}(\bar{N}^{\mathrm{H}}) + \frac{1}{R}\bar{W}_{,xx} = -\frac{1}{2}\tilde{L}(\bar{W} + 2\bar{W}^*, \bar{W}) \tag{2}$$

$$\tilde{L}_{31}(\bar{W}) + \tilde{L}_{32}(\bar{\Psi}_x) - \tilde{L}_{33}(\bar{\Psi}_y) + \tilde{L}_{34}(\bar{F}) - \tilde{L}_{35}(\bar{N}^{\mathrm{H}}) - \tilde{L}_{36}(\bar{S}^{\mathrm{H}}) = 0 \tag{3}$$

$$\tilde{L}_{41}(\bar{W}) - \tilde{L}_{42}(\bar{\Psi}_x) + \tilde{L}_{43}(\bar{\Psi}_y) + \tilde{L}_{44}(\bar{F}) - \tilde{L}_{45}(\bar{N}^{\mathrm{H}}) - \tilde{L}_{46}(\bar{S}^{\mathrm{H}}) = 0 \tag{4}$$

where linear operators $\tilde{L}_{ij}$ ( ) and nonlinear operator $\tilde{L}$ ( ) are defined in Appendix A.

The two end edges of the shell are assumed to be simply supported or clamped, so that the boundary conditions are $X = 0$, L:

$$\bar{W} = \bar{\Psi}_y = 0, \qquad \bar{M}_x = \bar{P}_x = 0 \text{ (simply supported)} \tag{5a}$$

$$\bar{W} = \bar{\Psi}_x = \bar{\Psi}_y = 0 \text{ (clamped)} \tag{5b}$$

$$\int_0^{2\pi R} \bar{N}_x \mathrm{d}Y + 2\pi R t \sigma_x + \pi R^2 q a = 0 \tag{5c}$$

where $a = 0$ and $a = 1$ for lateral and hydrostatic pressure loading case, respectively, and σ_x is the average axial compressive stress, $\bar{M}_x$ is the bending moment and $\bar{P}_x$ is higher order moment as defined in Reddy and Liu (1985). Also, we have the closed (or periodicity) condition

$$\int_0^{2\pi R} \frac{\partial \bar{V}}{\partial Y} \mathrm{d}Y = 0 \tag{6a}$$

or

$$\int_0^{2\pi R} \left[A_{22}^* \frac{\partial^2 \bar{F}}{\partial X^2} + A_{12}^* \frac{\partial^2 \bar{F}}{\partial Y^2} + \left(B_{21}^* - \frac{4}{3t^2} E_{21}^* \right) \frac{\partial \bar{\Psi}_x}{\partial X} + \left(B_{22}^* - \frac{4}{3t^2} E_{22}^* \right) \frac{\partial \bar{\Psi}_y}{\partial Y} - \frac{4}{3t^2} \left(E_{21}^* \frac{\partial^2 \bar{W}}{\partial X^2} + E_{22}^* \frac{\partial^2 \bar{W}}{\partial Y^2} \right) + \frac{\bar{W}}{R} - \frac{1}{2} \left(\frac{\partial \bar{W}}{\partial Y} \right)^2 - \frac{\partial \bar{W}}{\partial Y} \frac{\partial \bar{W}^*}{\partial Y} - \left(A_{12}^* \bar{N}_x^{\mathrm{H}} + A_{22}^* \bar{N}_y^{\mathrm{H}} \right) \right] \mathrm{d}Y = 0 \tag{6b}$$

Because of Eqs. (6a) and (6b) the in-plane boundary condition $\bar{V} = 0$ (at $X = 0$, L) is not needed in Eqs. (5a) and (5b).

The average end-shortening relationship is

$$\begin{aligned} \frac{\Delta_x}{L} &= -\frac{1}{2\pi R L} \int_0^{2\pi R} \int_0^L \frac{\partial \bar{U}}{\partial X} \mathrm{d}X \mathrm{d}Y \\ &= -\frac{1}{2\pi R L} \int_0^{2\pi R} \int_0^L \left[A_{11}^* \frac{\partial^2 \bar{F}}{\partial Y^2} + A_{12}^* \frac{\partial^2 \bar{F}}{\partial X^2} + \left(B_{11}^* - \frac{4}{3t^2} E_{11}^* \right) \frac{\partial \bar{\Psi}_x}{\partial X} + \left(B_{12}^* - \frac{4}{3t^2} E_{12}^* \right) \frac{\partial \bar{\Psi}_y}{\partial Y} \right. \\ &\quad \left. - \frac{4}{3t^2} \left(E_{11}^* \frac{\partial^2 \bar{W}}{\partial X^2} + E_{12}^* \frac{\partial^2 \bar{W}}{\partial Y^2} \right) - \frac{1}{2} \left(\frac{\partial \bar{W}}{\partial X} \right)^2 - \frac{\partial \bar{W}}{\partial X} \frac{\partial \bar{W}^*}{\partial X} - \left(A_{11}^* \bar{N}_x^{\mathrm{H}} + A_{12}^* \bar{N}_y^{\mathrm{H}} \right) \right] \mathrm{d}X \mathrm{d}Y \end{aligned} \tag{7}$$

The equivalent hygrothermal loads are defined as

$$\begin{bmatrix} \bar{N}^{\mathrm{H}} \\ \bar{M}^{\mathrm{H}} \\ \bar{S}^{\mathrm{H}} \end{bmatrix} = \begin{bmatrix} \bar{N}^{\mathrm{T}} \\ \bar{M}^{\mathrm{T}} \\ \bar{S}^{\mathrm{T}} \end{bmatrix} + \begin{bmatrix} \bar{N}^{\mathrm{m}} \\ \bar{M}^{\mathrm{m}} \\ \bar{S}^{\mathrm{m}} \end{bmatrix} \tag{8}$$

The forces, moments and higher order moments caused by elevated temperature or absorbed moisture are defined by

$$\begin{bmatrix} \bar{N}_x^{\mathrm{T}} & \bar{M}_x^{\mathrm{T}} & \bar{P}_x^{\mathrm{T}} \\ \bar{N}_y^{\mathrm{T}} & \bar{M}_y^{\mathrm{T}} & \bar{P}_y^{\mathrm{T}} \\ \bar{N}_{xy}^{\mathrm{T}} & \bar{M}_{xy}^{\mathrm{T}} & \bar{P}_{xy}^{\mathrm{T}} \end{bmatrix} = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (1, Z, Z^3) \begin{bmatrix} A_x \\ A_y \\ A_{xy} \end{bmatrix}_k \Delta T \, \mathrm{d}Z \tag{9a}$$

$$\begin{bmatrix} \bar{S}_x^{\mathrm{T}} \\ \bar{S}_y^{\mathrm{T}} \\ \bar{S}_{xy}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \bar{M}_x^{\mathrm{T}} \\ \bar{M}_y^{\mathrm{T}} \\ \bar{M}_{xy}^{\mathrm{T}} \end{bmatrix} - \frac{4}{3t^2} \begin{bmatrix} \bar{P}_x^{\mathrm{T}} \\ \bar{P}_y^{\mathrm{T}} \\ \bar{P}_{xy}^{\mathrm{T}} \end{bmatrix} \tag{9b}$$

and

$$\begin{bmatrix} \bar{N}_x^{\mathrm{m}} & \bar{M}_x^{\mathrm{m}} & \bar{P}_x^{\mathrm{m}} \\ \bar{N}_y^{\mathrm{m}} & \bar{M}_y^{\mathrm{m}} & \bar{P}_y^{\mathrm{m}} \\ \bar{N}_{xy}^{\mathrm{m}} & \bar{M}_{xy}^{\mathrm{m}} & \bar{P}_{xy}^{\mathrm{m}} \end{bmatrix} = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (1, Z, Z^3) \begin{bmatrix} B_x \\ B_y \\ B_{xy} \end{bmatrix}_k \Delta C \, \mathrm{d}Z \tag{9c}$$

$$\begin{bmatrix} \bar{S}_x^{\mathrm{m}} \\ \bar{S}_y^{\mathrm{m}} \\ \bar{S}_{xy}^{\mathrm{m}} \end{bmatrix} = \begin{bmatrix} \bar{M}_x^{\mathrm{m}} \\ \bar{M}_y^{\mathrm{m}} \\ \bar{M}_{xy}^{\mathrm{m}} \end{bmatrix} - \frac{4}{3t^2} \begin{bmatrix} \bar{P}_x^{\mathrm{m}} \\ \bar{P}_y^{\mathrm{m}} \\ \bar{P}_{xy}^{\mathrm{m}} \end{bmatrix} \tag{9d}$$

where ΔT is temperature rise from some reference temperature at which there are no thermal strains and ΔC is the increase from zero moisture measured in terms of the percentage weight increase, and

$$\begin{bmatrix} A_x \\ A_y \\ A_{xy} \end{bmatrix} = -\begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 \\ s^2 & c^2 \\ 2cs & -2cs \end{bmatrix} \begin{bmatrix} \alpha_{11} \\ \alpha_{22} \end{bmatrix} \tag{10a}$$

$$\begin{bmatrix} B_x \\ B_y \\ B_{xy} \end{bmatrix} = -\begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 \\ s^2 & c^2 \\ 2cs & -2cs \end{bmatrix} \begin{bmatrix} \beta_{11} \\ \beta_{22} \end{bmatrix} \tag{10b}$$

where $\bar{Q}_{ij}$ are the transformed elastic constants, defined by

$$\begin{bmatrix} \bar{Q}_{11} \\ \bar{Q}_{12} \\ \bar{Q}_{22} \\ \bar{Q}_{16} \\ \bar{Q}_{26} \\ \bar{Q}_{66} \end{bmatrix} = \begin{bmatrix} c^4 & 2c^2s^2 & s^4 & 4c^2s^2 \\ c^2s^2 & c^4+s^4 & c^2s^2 & -4c^2s^2 \\ s^4 & 2c^2s^2 & c^4 & 4c^2s^2 \\ c^3s & cs^3-c^3s & -cs^3 & -2cs(c^2-s^2) \\ cs^3 & c^3s-cs^3 & -c^3s & 2cs(c^2-s^2) \\ c^2s^2 & -2c^2s^2 & c^2s^2 & (c^2-s^2)^2 \end{bmatrix} \begin{bmatrix} Q_{11} \\ Q_{12} \\ Q_{22} \\ Q_{66} \end{bmatrix} \tag{11a}$$

$$\begin{bmatrix} \bar{Q}_{44} \\ \bar{Q}_{45} \\ \bar{Q}_{55} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 \\ -cs & cs \\ s^2 & c^2 \end{bmatrix} \begin{bmatrix} Q_{44} \\ Q_{55} \end{bmatrix} \tag{11b}$$

where

$$Q_{11}=\frac{E_{11}}{(1-\nu_{12}\nu_{21})},\quad Q_{22}=\frac{E_{22}}{(1-\nu_{12}\nu_{21})},\quad Q_{12}=\frac{\nu_{21}E_{11}}{(1-\nu_{12}\nu_{21})},$$

$$Q_{44}=G_{23},\quad Q_{55}=G_{13},\quad Q_{66}=G_{12} \tag{11c}$$

and

$$c=\cos\theta,\quad s=\sin\theta \tag{11d}$$

where θ is the lamination angle with respect to the shell X-axis.

In terms of a micro-mechanical model of the laminate, the thermal expansion coefficients in the longitudinal and transverse directions may be written as (see Tsai and Hahn, 1980)

$$\alpha_{11}=\frac{V_f E_f \alpha_f + V_m E_m \alpha_m}{V_f E_f + V_m E_m} \tag{12a}$$

$$\alpha_{22}=(1+\nu_f)V_f\alpha_f+(1+\nu_m)V_m\alpha_m-\nu_{12}\alpha_{11} \tag{12b}$$

where α_f and α_m are thermal expansion coefficients of the fibre and matrix respectively and the longitudinal and transverse coefficients of hygroscopic expansion of a lamina may be written as

$$\beta_{11}=\frac{V_f E_f c_{fm}\beta_f + V_m E_m \beta_m}{E_{11}(V_f\rho_f c_{fm}+V_m\rho_m)}\rho \tag{13a}$$

$$\beta_{22}=\frac{V_f(1+\nu_f)c_{fm}\beta_f+V_m(1+\nu_m)\beta_m}{V_f\rho_f c_{fm}+V_m\rho_m}\rho-\nu_{12}\beta_{11} \tag{13b}$$

where c_{fm} is the moisture concentration ratio, β_f and β_m are the swelling coefficients of the fibre and matrix, and ρ, ρ_f and ρ_m are mass densities of a lamina, fibre and matrix respectively and are related by

$$\rho=V_f\rho_f+V_m\rho_m \tag{14}$$

In the above equations, V_f and V_m are the fibre and matrix volume fractions and are related by

$$V_f+V_m=1 \tag{15}$$

and E_f, G_f and ν_f are the Young's modulus, shear modulus and Poisson's ratio respectively of the fibre, and E_m, G_m and ν_m are corresponding properties for the matrix.

$$E_{11}=V_f E_f+V_m E_m \tag{16a}$$

$$\frac{1}{E_{22}}=\frac{V_f}{E_f}+\frac{V_m}{E_m}-V_f V_m\frac{\nu_f^2 E_m/E_f+\nu_m^2 E_f/E_m-2\nu_f\nu_m}{V_f E_f+V_m E_m} \tag{16b}$$

$$\frac{1}{G_{12}}=\frac{V_f}{G_f}+\frac{V_m}{G_m} \tag{16c}$$

$$\nu_{12}=V_f\nu_f+V_m\nu_m \tag{16d}$$

It is assumed that E_m is a function of temperature and moisture, as shown in Section 4, so that α_{11}, α_{22}, β_{11}, β_{22}, E_{11}, E_{22}, G_{12}, $G_{13}(=G_{12})$ and $G_{23}(=0.5\,G_{12})$ are also functions of temperature and moisture. Furthermore, in Eqs. (6b) and (7), and Eq. (19) below, the reduced stiffness matrices $[A^*_{ij}]$, $[B^*_{ij}]$, $[D^*_{ij}]$, $[E^*_{ij}]$, $[F^*_{ij}]$ and $[H^*_{ij}]$ $(i,j=1,2,6)$ are functions of temperature and moisture, defined by

$$\mathbf{A}^* = \mathbf{A}^{-1}, \quad \mathbf{B}^* = -\mathbf{A}^{-1}\mathbf{B}, \quad \mathbf{D}^* = \mathbf{D} - \mathbf{B}\mathbf{A}^{-1}\mathbf{B}, \quad \mathbf{E}^* = -\mathbf{A}^{-1}\mathbf{E}, \quad \mathbf{F}^* = \mathbf{F} - \mathbf{E}\mathbf{A}^{-1}\mathbf{B},$$
$$\mathbf{H}^* = \mathbf{H} - \mathbf{E}\mathbf{A}^{-1}\mathbf{E} \tag{17}$$

where A_{ij}, B_{ij} etc., are the laminate stiffnesses, defined by

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (\bar{Q}_{ij})_k (1, Z, Z^2, Z^3, Z^4, Z^6)\, \mathrm{d}Z \quad (i,j = 1,2,6) \tag{18a}$$

$$(A_{ij}, D_{ij}, F_{ij}) = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (\bar{Q}_{ij})_k (1, Z^2, Z^4)\, \mathrm{d}Z \quad (i,j = 4,5) \tag{18b}$$

3. Analytical method and asymptotic solutions

Having developed the theory, we now try to solve Eqs. (1)–(4) with boundary conditions (5a)–(5c). Before proceeding, it is convenient first to define the following dimensionless quantities (with γ_{ijk} in Eqs. (25), (27) and (28) below are defined as in Appendix B)

$$\begin{aligned}
&x = \pi X/L, \quad y = Y/R, \quad \beta = L/\pi R, \quad \bar{Z} = L^2/Rt, \quad \varepsilon = (\pi^2 R/L^2)[D_{11}^* D_{22}^* A_{11}^* A_{22}^*]^{1/4},\\
&(W, W^*) = \varepsilon(\bar{W}, \bar{W}^*)/[D_{11}^* D_{22}^* A_{11}^* A_{22}^*]^{1/4}, \qquad F = \varepsilon^2 \bar{F}/[D_{11}^* D_{22}^*]^{1/2},\\
&(\Psi_x, \Psi_y) = \varepsilon^2(\bar{\Psi}_x, \bar{\Psi}_y)(L/\pi)/[D_{11}^* D_{22}^* A_{11}^* A_{22}^*]^{1/4},\\
&\gamma_{14} = [D_{22}^*/D_{11}^*]^{1/2}, \quad \gamma_{24} = [A_{11}^*/A_{22}^*]^{1/2}, \quad \gamma_5 = -A_{12}^*/A_{22}^*,\\
&(\gamma_{31}, \gamma_{41}) = (L^2/\pi^2)(A_{55} - 8D_{55}/t^2 + 16F_{55}/t^4, \qquad A_{44} - 8D_{44}/t^2 + 16F_{44}/t^4)/D_{11}^*\\
&(\gamma_{T1}, \gamma_{T2}, \gamma_{m1}, \gamma_{m2}) = (A_x^{\mathrm{T}}, A_y^{\mathrm{T}}, B_x^{\mathrm{m}}, B_y^{\mathrm{m}})R/\left[D_{11}^* D_{22}^*/A_{11}^* A_{22}^*\right]^{1/4},\\
&(M_x, P_x) = \varepsilon^2(\bar{M}_x, 4\bar{P}_x/3t^2)L^2/\pi^2 D_{11}^*\left[D_{11}^* D_{22}^* A_{11}^* A_{22}^*\right]^{1/4},\\
&\lambda_p = \sigma_x/(2/Rt)\left[D_{11}^* D_{22}^*/A_{11}^* A_{22}^*\right]^{1/4}, \qquad \delta_p = (\Delta_x/L)/(2/R)\left[D_{11}^* D_{22}^* A_{11}^* A_{22}^*\right]^{1/4},\\
&\lambda_q = q(3)^{3/4} L R^{3/2}[A_{11}^* A_{22}^*]^{1/8}/4\pi[D_{11}^* D_{22}^*]^{3/8}, \quad \delta_q = (\Delta_x/L)(3)^{3/4} L R^{1/2}/4\pi\left[D_{11}^* D_{22}^* A_{11}^* A_{22}^*\right]^{3/8}
\end{aligned} \tag{19}$$

and let

$$\begin{bmatrix} A_x^{\mathrm{T}} & B_x^{\mathrm{m}} \\ A_y^{\mathrm{T}} & B_y^{\mathrm{m}} \end{bmatrix} = -\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \begin{bmatrix} A_x & B_x \\ A_y & B_y \end{bmatrix}_k \mathrm{d}Z \tag{20}$$

The nonlinear Eqs. (1)–(4) may then be written in dimensionless form as

$$\varepsilon^2 L_{11}(W) - \varepsilon L_{12}(\Psi_x) - \varepsilon L_{13}(\Psi_y) + \varepsilon\gamma_{14} L_{14}(F) - \gamma_{14} F_{,xx} = \gamma_{14}\beta^2 L(W + W^*, F) + \gamma_{14}\tfrac{4}{3}(3)^{1/4}\lambda_q \varepsilon^{3/2} \tag{21}$$

$$L_{21}(F) + \gamma_{24} L_{22}(\Psi_x) + \gamma_{24} L_{23}(\Psi_y) - \varepsilon\gamma_{24} L_{24}(W) + \gamma_{24} W_{,xx} = -\tfrac{1}{2}\gamma_{24}\beta^2 L(W + 2W^*, W) \tag{22}$$

$$\varepsilon L_{31}(W) + L_{32}(\Psi_x) - L_{33}(\Psi_y) + \gamma_{14} L_{34}(F) = 0 \tag{23}$$

$$\varepsilon L_{41}(W) - L_{42}(\Psi_x) + L_{43}(\Psi_y) + \gamma_{14} L_{44}(F) = 0 \tag{24}$$

where

$$\begin{aligned}
L_{11}(\ ) &= \gamma_{110}\frac{\partial^4}{\partial x^4} + 2\gamma_{112}\beta^2\frac{\partial^4}{\partial x^2\partial y^2} + \gamma_{114}\beta^4\frac{\partial^4}{\partial y^4}\\
L_{12}(\ ) &= \gamma_{120}\frac{\partial^3}{\partial x^3} + \gamma_{122}\beta^2\frac{\partial^3}{\partial x\partial y^2}\\
L_{13}(\ ) &= \gamma_{131}\beta\frac{\partial^3}{\partial x^2\partial y} + \gamma_{133}\beta^3\frac{\partial^3}{\partial y^3}\\
L_{14}(\ ) &= \gamma_{140}\frac{\partial^4}{\partial x^4} + 2\gamma_{142}\beta^2\frac{\partial^4}{\partial x^2\partial y^2} + \gamma_{144}\beta^4\frac{\partial^4}{\partial y^4}\\
L_{21}(\ ) &= \frac{\partial^4}{\partial x^4} + 2\gamma_{212}\beta^2\frac{\partial^4}{\partial x^2\partial y^2} + \gamma_{214}\beta^4\frac{\partial^4}{\partial y^4}\\
L_{22}(\ ) &= \gamma_{220}\frac{\partial^3}{\partial x^3} + \gamma_{222}\beta^2\frac{\partial^3}{\partial x\partial y^2}\\
L_{23}(\ ) &= \gamma_{231}\beta\frac{\partial^3}{\partial x^2\partial y} + \gamma_{233}\beta^3\frac{\partial^3}{\partial y^3}\\
L_{24}(\ ) &= \gamma_{240}\frac{\partial^4}{\partial x^4} + 2\gamma_{242}\beta^2\frac{\partial^4}{\partial x^2\partial y^2} + \gamma_{244}\beta^4\frac{\partial^4}{\partial y^4}\\
L_{31}(\ ) &= \gamma_{31}\frac{\partial}{\partial x} + \gamma_{310}\frac{\partial^3}{\partial x^3} + \gamma_{312}\beta^2\frac{\partial^3}{\partial x\partial y^2}\\
L_{32}(\ ) &= \gamma_{31} - \gamma_{320}\frac{\partial^2}{\partial x^2} - \gamma_{322}\beta^2\frac{\partial^2}{\partial y^2}\\
L_{33}(\ ) &= \gamma_{331}\beta\frac{\partial^2}{\partial x\partial y}\\
L_{34}(\ ) &= L_{22}(\ )\\
L_{41}(\ ) &= \gamma_{41}\beta\frac{\partial}{\partial y} + \gamma_{411}\beta\frac{\partial^3}{\partial x^2\partial y} + \gamma_{413}\beta^3\frac{\partial^3}{\partial y^3}\\
L_{42}(\ ) &= L_{33}(\ )\\
L_{43}(\ ) &= \gamma_{41} - \gamma_{430}\frac{\partial^2}{\partial x^2} - \gamma_{432}\beta^2\frac{\partial^2}{\partial y^2}\\
L_{44}(\ ) &= L_{23}(\ )\\
L(\ ) &= \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} - 2\frac{\partial^2}{\partial x\partial y}\frac{\partial^2}{\partial x\partial y} + \frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2}
\end{aligned} \tag{25}$$

Because of the definition of ε given in Eq. (19), for most of the composite materials $\left[D^*_{11}D^*_{22}A^*_{11}A^*_{22}\right]^{1/4} = (0.2-0.3)t$, hence when $\bar{Z} = (L^2/Rt) > 2.96$, we have $\varepsilon < 1$. Specially, for isotropic cylindrical shells, we have $\varepsilon = \pi^2/\bar{Z}_B\sqrt{12}$, where $\bar{Z}_B = (L^2/Rt)[1-\nu^2]^{1/2}$ is the Batdorf shell parameter, which should be greater than 2.85 in the case of classical linear buckling analysis (Batdorf, 1947). In practice, the shell structure will have $\bar{Z} \geqslant 10$, so that we always have $\varepsilon \ll 1$. When $\varepsilon < 1$, then Eqs. (21)–(24) are the equations of the boundary layer type, from which nonlinear prebuckling deformations, large deflections in the postbuckling range and initial geometric imperfections of the shell can be considered simultaneously. The boundary conditions of Eqs. (5a)–(5c) become $x = 0, \pi$:

$$W = \Psi_y = 0, \quad M_x = P_x = 0 \text{ (simply supported)} \tag{26a}$$

$$W = \Psi_x = \Psi_y = 0 \text{ (clamped)} \tag{26b}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \beta^2 \frac{\partial^2 F}{\partial y^2}\,\mathrm{d}y + 2\lambda_p \varepsilon + \frac{2}{3}(3)^{1/4}\lambda_q \varepsilon^{3/2} a = 0 \tag{26c}$$

and the closed condition becomes

$$\int_0^{2\pi}\left[\left(\frac{\partial^2 F}{\partial x^2} - \gamma_5\beta^2\frac{\partial^2 F}{\partial y^2}\right) + \gamma_{24}\left(\gamma_{220}\frac{\partial \Psi_x}{\partial x} + \gamma_{522}\beta\frac{\partial \Psi_y}{\partial y}\right) - \varepsilon\gamma_{24}\left(\gamma_{240}\frac{\partial^2 W}{\partial x^2} + \gamma_{622}\beta^2\frac{\partial^2 W}{\partial y^2}\right) + \gamma_{24}W - \frac{1}{2}\gamma_{24}\beta^2\left(\frac{\partial W}{\partial y}\right)^2 - \gamma_{24}\beta^2\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y} + \varepsilon(\gamma_{T2} - \gamma_5\gamma_{T1})\Delta T + \varepsilon(\gamma_{\mathrm{m}2} - \gamma_5\gamma_{\mathrm{m}1})\Delta C\right]\mathrm{d}y = 0 \tag{27}$$

In this section two loading conditions will be considered, so that the unit end-shortening relationship may be written in two dimensionless forms as

$$\delta_q = -\frac{(3)^{3/4}}{8\pi^2\gamma_{24}}\varepsilon^{-3/2}\int_0^{2\pi}\int_0^{\pi}\left[\left(\gamma_{24}^2\beta^2\frac{\partial^2 F}{\partial y^2} - \gamma_5\frac{\partial^2 F}{\partial x^2}\right) + \gamma_{24}\left(\gamma_{511}\frac{\partial \Psi_x}{\partial x} + \gamma_{233}\beta\frac{\partial \Psi_y}{\partial y}\right) - \varepsilon\gamma_{24}\left(\gamma_{611}\frac{\partial^2 W}{\partial x^2} + \gamma_{244}\beta^2\frac{\partial^2 W}{\partial y^2}\right) - \frac{1}{2}\gamma_{24}\left(\frac{\partial W}{\partial x}\right)^2 - \gamma_{24}\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x} + \varepsilon(\gamma_{24}^2\gamma_{T1} - \gamma_5\gamma_{T2})\Delta T + \varepsilon(\gamma_{24}^2\gamma_{\mathrm{m}1} - \gamma_5\gamma_{\mathrm{m}2})\Delta C\right]\mathrm{d}x\,\mathrm{d}y \tag{28a}$$

$$\delta_p = -\frac{1}{4\pi^2\gamma_{24}}\varepsilon^{-1}\int_0^{2\pi}\int_0^{\pi}\left[\left(\gamma_{24}^2\beta^2\frac{\partial^2 F}{\partial y^2} - \gamma_5\frac{\partial^2 F}{\partial x^2}\right) + \gamma_{24}\left(\gamma_{511}\frac{\partial \Psi_x}{\partial x} + \gamma_{233}\beta\frac{\partial \Psi_y}{\partial y}\right) - \varepsilon\gamma_{24}\left(\gamma_{611}\frac{\partial^2 W}{\partial x^2} + \gamma_{244}\beta^2\frac{\partial^2 W}{\partial y^2}\right) - \frac{1}{2}\gamma_{24}\left(\frac{\partial W}{\partial x}\right)^2 - \gamma_{24}\frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x} + \varepsilon(\gamma_{24}^2\gamma_{T1} - \gamma_5\gamma_{T2})\Delta T + \varepsilon(\gamma_{24}^2\gamma_{\mathrm{m}1} - \gamma_5\gamma_{\mathrm{m}2})\Delta C\right]\mathrm{d}x\,\mathrm{d}y \tag{28b}$$

By virtue of the fact that ΔC and ΔT are assumed to be uniform, the hygrothermal coupling in Eqs. (1)–(4) vanishes, but terms in ΔC and ΔT intervene in Eqs. (27), (28a) and (28b).

Applying Eqs. (21)–(28b), the postbuckling behaviour of perfect and imperfect, shear deformable laminated cylindrical shells subjected to combined mechanical loading under hygrothermal environments is determined by a singular perturbation technique. The essence of this procedure, in the present case, is to assume that

$$\begin{aligned}
W &= w(x,y,\varepsilon) + \tilde{W}(x,\xi,y,\varepsilon) + \hat{W}(x,\varsigma,y,\varepsilon)\\
F &= f(x,y,\varepsilon) + \tilde{F}(x,\xi,y,\varepsilon) + \hat{F}(x,\varsigma,y,\varepsilon)\\
\Psi_x &= \psi_x(x,y,\varepsilon) + \tilde{\Psi}_x(x,\xi,y,\varepsilon) + \hat{\Psi}_x(x,\varsigma,y,\varepsilon)\\
\Psi_y &= \psi_y(x,y,\varepsilon) + \tilde{\Psi}_y(x,\xi,y,\varepsilon) + \hat{\Psi}_y(x,\varsigma,y,\varepsilon)
\end{aligned} \tag{29}$$

where ε is a small perturbation parameter (see beneath Eq. (25)) and $w(x,y,\varepsilon)$, $f(x,y,\varepsilon)$ are called outer solutions or regular solutions of the shell, $\tilde{W}(x,\xi,y,\varepsilon)$, $\tilde{F}(x,\xi,y,\varepsilon)$, $\tilde{\Psi}_x(x,\xi,y,\varepsilon)$, $\tilde{\Psi}_y(x,\xi,y,\varepsilon)$ and $\hat{W}(x,\varsigma,y,\varepsilon)$, $\hat{F}(x,\varsigma,y,\varepsilon)$, $\hat{\Psi}_x(x,\varsigma,y,\varepsilon)$, $\hat{\Psi}_y(x,\varsigma,y,\varepsilon)$ are the boundary layer solutions near the $x = 0$ and $x = \pi$ edges, respectively, and ξ and ζ are the boundary layer variables, defined as

$$\xi = x/\sqrt{\varepsilon}, \quad \varsigma = (\pi - x)/\sqrt{\varepsilon} \tag{30}$$

(This means for isotropic cylindrical shells the width of the boundary layers is of the order $\sqrt{Rt}$.) In Eq. (29) the regular and boundary layer solutions are taken in the form of perturbation expansions as

$$\begin{aligned} w(x,y,\varepsilon) &= \sum_{j=1} \varepsilon^{j/2} w_{j/2}(x,y), \quad f(x,y,\varepsilon) = \sum_{j=0} \varepsilon^{j/2} f_{j/2}(x,y) \\ \psi_x(x,y,\varepsilon) &= \sum_{j=1} \varepsilon^{j/2} (\psi_x)_{j/2}(x,y), \quad \psi_y(x,y,\varepsilon) = \sum_{j=1} \varepsilon^{j/2} (\psi_y)_{j/2}(x,y) \end{aligned} \tag{31a}$$

$$\begin{aligned} \tilde{W}(x,\xi,y,\varepsilon) &= \sum_{j=0} \varepsilon^{j/2+1} \tilde{W}_{j/2+1}(x,\xi,y), \quad \tilde{F}(x,\xi,y,\varepsilon) = \sum_{j=0} \varepsilon^{j/2+2} \tilde{F}_{j/2+2}(x,\xi,y) \\ \tilde{\Psi}_x(x,\xi,y,\varepsilon) &= \sum_{j=0} \varepsilon^{(j+3)/2} (\tilde{\Psi}_x)_{(j+3)/2}(x,\xi,y), \quad \tilde{\Psi}_y(x,\xi,y,\varepsilon) = \sum_{j=0} \varepsilon^{j/2+2} (\tilde{\Psi}_y)_{j/2+2}(x,\xi,y) \end{aligned} \tag{31b}$$

$$\begin{aligned} \hat{W}(x,\varsigma,y,\varepsilon) &= \sum_{j=0} \varepsilon^{j/2+1} \hat{W}_{j/2+1}(x,\varsigma,y), \quad \hat{F}(x,\varsigma,y,\varepsilon) = \sum_{j=0} \varepsilon^{j/2+2} \hat{F}_{j/2+2}(x,\varsigma,y) \\ \hat{\Psi}_x(x,\varsigma,y,\varepsilon) &= \sum_{j=0} \varepsilon^{(j+3)/2} (\hat{\Psi}_x)_{(j+3)/2}(x,\varsigma,y), \quad \hat{\Psi}_y(x,\varsigma,y,\varepsilon) = \sum_{j=0} \varepsilon^{j/2+2} (\hat{\Psi}_y)_{j/2+2}(x,\varsigma,y) \end{aligned} \tag{31c}$$

The initial buckling mode is assumed to have the form

$$w_2(x,y) = A_{11}^{(2)} \sin mx \sin ny \tag{32}$$

It should be remembered that, because of the definition of W given in Eq. (19), this means that $w_2(x,y)$ corresponds to $\bar{w}_1(X,Y)$ and the initial geometric imperfection is assumed to have a similar form

$$W^*(x,y,\varepsilon) = \varepsilon^2 a_{11}^* \sin mx \sin ny = \varepsilon^2 \mu A_{11}^{(2)} \sin mx \sin ny \tag{33}$$

where $\mu = a_{11}^*/A_{11}^{(2)}$ is the imperfection parameter.

Substituting Eqs. (29)–(31c) into Eqs. (21)–(24), collecting the terms of the same order of ε, three sets of perturbation equations are obtained for the regular and boundary layer solutions, respectively. It has been shown (Shen and Chen, 1988, 1990) that the effect of the boundary layer on the buckling load of the shell under axial compression is quite different from that of the shell subjected to external pressure. To this end, two kinds of loading conditions will be considered.

Case (1) high values of external pressure combined with relatively low axial load. Let

$$\frac{P_0}{\pi R^2 q} = b_1 \tag{34a}$$

or

$$\frac{2\lambda_p \varepsilon}{\frac{4}{3}(3)^{1/4} \lambda_q \varepsilon^{3/2}} = \frac{b_1}{2} \tag{34b}$$

In this case, the boundary condition of Eq. (26c) becomes

$$\frac{1}{2\pi} \int_0^{2\pi} \beta^2 \frac{\partial^2 F}{\partial y^2} \mathrm{d}y + \frac{2}{3}(3)^{1/4} \lambda_q \varepsilon^{3/2}(a + b_1) = 0 \tag{35}$$

For convenience we replace $(a + b_1)$ with a_1 in Eq. (37) below, by using Eqs. (32) and (33) to solve these perturbation equations of each order, and matching the regular solutions with the boundary layer solutions at the each end of the shell, so that the asymptotic solutions satisfying the clamped boundary conditions are constructed as

$$W = \varepsilon^{3/2}\left[A_{00}^{(3/2)} - A_{00}^{(3/2)}\left(a_{01}^{(3/2)}\cos\phi\frac{x}{\sqrt{\varepsilon}} + a_{10}^{(3/2)}\sin\phi\frac{x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{x}{\sqrt{\varepsilon}}\right) - A_{00}^{(3/2)}\left(a_{01}^{(3/2)}\cos\phi\frac{\pi - x}{\sqrt{\varepsilon}} + a_{10}^{(3/2)}\sin\phi\frac{\pi - x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi - x}{\sqrt{\varepsilon}}\right)\right] + \varepsilon^2\left[A_{11}^{(2)}\sin mx\sin ny\right] + \varepsilon^3\left[A_{11}^{(3)}\sin mx\sin ny\right] + \varepsilon^4\left[A_{00}^{(4)} + A_{20}^{(4)}\cos 2mx + A_{02}^{(4)}\cos 2ny\right] + O(\varepsilon^5) \tag{36}$$

$$F = -\frac{1}{2}B_{00}^{(0)}\left(\beta^2x^2 + a_1\frac{y^2}{2}\right) + \varepsilon\left[-\frac{1}{2}B_{00}^{(1)}\left(\beta^2x^2 + a_1\frac{y^2}{2}\right)\right] + \varepsilon^2\left[-\frac{1}{2}B_{00}^{(2)}\left(\beta^2x^2 + a_1\frac{y^2}{2}\right) + B_{11}^{(2)}\sin mx\sin ny\right] + \varepsilon^{5/2}\left[A_{00}^{(3/2)}\left(b_{01}^{(5/2)}\cos\phi\frac{x}{\sqrt{\varepsilon}} + b_{10}^{(5/2)}\sin\phi\frac{x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{x}{\sqrt{\varepsilon}}\right) + A_{00}^{(3/2)}\left(b_{01}^{(5/2)}\cos\phi\frac{\pi - x}{\sqrt{\varepsilon}} + b_{10}^{(5/2)}\sin\phi\frac{\pi - x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi - x}{\sqrt{\varepsilon}}\right)\right] + \varepsilon^3\left[-\frac{1}{2}B_{00}^{(3)}\left(\beta^2x^2 + a_1\frac{y^2}{2}\right)\right] + \varepsilon^4\left[-\frac{1}{2}B_{00}^{(4)}\left(\beta^2x^2 + a_1\frac{y^2}{2}\right) + B_{11}^{(4)}\sin mx\sin ny + B_{20}^{(4)}\cos 2mx + B_{02}^{(4)}\cos 2ny\right] + O(\varepsilon^5) \tag{37}$$

$$\Psi_x = \varepsilon^2\left[C_{11}^{(2)}\cos mx\sin ny + \left(c_{01}^{(2)}\cos\phi\frac{x}{\sqrt{\varepsilon}} + c_{10}^{(2)}\sin\phi\frac{x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{x}{\sqrt{\varepsilon}}\right) + \left(c_{01}^{(2)}\cos\phi\frac{\pi - x}{\sqrt{\varepsilon}} + c_{10}^{(2)}\sin\phi\frac{\pi - x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi - x}{\sqrt{\varepsilon}}\right)\right] + \varepsilon^3\left[C_{11}^{(3)}\cos mx\sin ny\right] + \varepsilon^4\left[C_{11}^{(4)}\cos mx\sin ny + C_{20}^{(4)}\sin 2mx\right] + O(\varepsilon^5) \tag{38}$$

$$\Psi_y = \varepsilon^2\left[D_{11}^{(2)}\sin mx\cos ny\right] + \varepsilon^3\left[D_{11}^{(3)}\sin mx\cos ny\right] + \varepsilon^4\left[D_{11}^{(4)}\sin mx\cos ny + D_{02}^{(4)}\sin 2ny\right] + O(\varepsilon^5) \tag{39}$$

Note that all coefficients in Eqs. (36)–(39) are related and can be written as functions of $A_{11}^{(2)}$, but for the sake of brevity the detailed expressions are not shown, whereas α and ϕ are given in detail in Appendix C.

Next, substituting Eqs. (36)–(39) into the boundary condition (35) and into Eq. (28a), the postbuckling equilibrium paths can be written as

$$\lambda_q = \lambda_q^{(0)} + \lambda_q^{(2)}(A_{11}^{(2)}\varepsilon^2)^2 + \cdots \tag{40}$$

and

$$\delta_q = \delta_q^{(0)} - \delta_q^{(H)} + \delta_q^{(2)}(A_{11}^{(2)}\varepsilon^2)^2 + \cdots \tag{41}$$

in Eqs. (40) and (41), $(A_{11}^{(2)}\varepsilon^2)$ is taken as the second perturbation parameter relating to the dimensionless maximum deflection. If the maximum deflection is assumed to be at the point $(x, y) = (\pi/2m, \pi/2n)$, then

$$A_{11}^{(2)}\varepsilon^2 = W_{\mathrm{m}} - \Theta_1 W_{\mathrm{m}}^2 + \cdots \tag{42a}$$

and the dimensionless maximum deflection of the shell is written as

$$W_{\mathrm{m}} = \frac{1}{C_3}\left[\varepsilon\frac{t}{[D_{11}^*D_{22}^*A_{11}^*A_{22}^*]^{1/4}}\frac{\bar{W}}{t} + \Theta_2\right] \tag{42b}$$

All symbols used in Eqs. (40)–(42b) and Eqs. (49)–(51b) below are also described in detail in Appendix C.

Case (2) high values of axial compression combined with relatively low external pressure. Let

$$\frac{\pi R^2 q}{P_0} = b_2 \tag{43a}$$

or

$$\frac{\frac{4}{3}(3)^{1/4}\lambda_q \varepsilon^{3/2}}{2\lambda_p \varepsilon} = 2b_2 \tag{43b}$$

In this case, the boundary condition of Eq. (26c) becomes

$$\frac{1}{2\pi}\int_0^{2\pi} \beta^2 \frac{\partial^2 F}{\partial y^2}\,\mathrm{d}y + 2\lambda_p \varepsilon(1 + ab_2) = 0 \tag{44}$$

Similarly, by taking $a_2 = 2b_2/(1 + ab_2)$ and using a singular perturbation procedure, the asymptotic solutions satisfying the clamped boundary conditions are obtained as

$$\begin{aligned} W = {} & \varepsilon\Bigg[A_{00}^{(1)} - A_{00}^{(1)}\left(a_{01}^{(1)}\cos\phi\frac{x}{\sqrt{\varepsilon}} + a_{10}^{(1)}\sin\phi\frac{x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{x}{\sqrt{\varepsilon}}\right) - A_{00}^{(1)}\left(a_{01}^{(1)}\cos\phi\frac{\pi - x}{\sqrt{\varepsilon}}\right. \\ & \left.+ a_{10}^{(1)}\sin\phi\frac{\pi - x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi - x}{\sqrt{\varepsilon}}\right)\Bigg] + \varepsilon^2\Bigg[A_{11}^{(2)}\sin mx\sin ny + A_{02}^{(2)}\cos 2ny - \left(A_{02}^{(2)}\cos 2ny\right) \\ & \times\left(a_{01}^{(1)}\cos\phi\frac{x}{\sqrt{\varepsilon}}. + a_{10}^{(1)}\sin\phi\frac{x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{x}{\sqrt{\varepsilon}}\right) - \left(A_{02}^{(2)}\cos 2ny\right) \\ & \times\left(a_{01}^{(1)}\cos\phi\frac{\pi - x}{\sqrt{\varepsilon}} + a_{10}^{(1)}\sin\phi\frac{\pi - x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi - x}{\sqrt{\varepsilon}}\right)\Bigg] + \varepsilon^3\left[A_{11}^{(3)}\sin mx\sin ny + A_{02}^{(3)}\cos 2ny\right] \\ & + \varepsilon^4\left[A_{00}^{(4)} + A_{20}^{(4)}\cos 2mx + A_{02}^{(4)}\cos 2ny + A_{13}^{(4)}\sin mx\sin 3ny + A_{04}^{(4)}\cos 4ny\right] + O(\varepsilon^5) \end{aligned} \tag{45}$$

$$\begin{aligned} F = {} & -\frac{1}{2}B_{00}^{(0)}(a_2\beta^2x^2 + y^2) + \varepsilon\left[-\frac{1}{2}B_{00}^{(1)}(a_2\beta^2x^2 + y^2)\right] + \varepsilon^2\Bigg[-\frac{1}{2}B_{00}^{(2)}(a_2\beta^2x^2 + y^2) \\ & + B_{11}^{(2)}\sin mx\sin ny + A_{00}^{(1)}\left(b_{01}^{(2)}\cos\phi\frac{x}{\sqrt{\varepsilon}} + b_{10}^{(2)}\sin\phi\frac{x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{x}{\sqrt{\varepsilon}}\right) \\ & + A_{00}^{(1)}\left(b_{01}^{(2)}\cos\phi\frac{\pi - x}{\sqrt{\varepsilon}} + b_{10}^{(2)}\sin\phi\frac{\pi - x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi - x}{\sqrt{\varepsilon}}\right)\Bigg] + \varepsilon^3\Bigg[-\frac{1}{2}B_{00}^{(3)}(a_2\beta^2x^2 + y^2) \\ & + B_{02}^{(3)}\cos 2ny + (A_{02}^{(2)}\cos 2ny)\left(b_{01}^{(3)}\cos\phi\frac{x}{\sqrt{\varepsilon}} + b_{10}^{(3)}\sin\phi\frac{x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{x}{\sqrt{\varepsilon}}\right) + (A_{02}^{(2)}\cos 2ny) \\ & \times\left(b_{01}^{(3)}\cos\phi\frac{\pi - x}{\sqrt{\varepsilon}} + b_{10}^{(3)}\sin\phi\frac{\pi - x}{\sqrt{\varepsilon}}\right)\exp\left(-\alpha\frac{\pi - x}{\sqrt{\varepsilon}}\right)\Bigg] + \varepsilon^4\Bigg[-\frac{1}{2}B_{00}^{(4)}(a_2\beta^2x^2 + y^2) \\ & + B_{11}^{(4)}\sin mx\sin ny + B_{20}^{(4)}\cos 2mx + B_{02}^{(4)}\cos 2ny + B_{13}^{(4)}\sin mx\sin 3ny\Bigg] + O(\varepsilon^5) \end{aligned} \tag{46}$$

$$\begin{aligned} \Psi_x = {} & \varepsilon^{3/2}\left[A_{00}^{(1)}c_{10}^{(3/2)}\sin\phi\frac{x}{\sqrt{\varepsilon}}\exp\left(-\alpha\frac{x}{\sqrt{\varepsilon}}\right) + A_{00}^{(1)}c_{10}^{(3/2)}\sin\phi\frac{\pi - x}{\sqrt{\varepsilon}}\exp\left(-\alpha\frac{\pi - x}{\sqrt{\varepsilon}}\right)\right] \\ & + \varepsilon^2[C_{11}^{(2)}\cos mx\sin ny] + \varepsilon^{5/2}\Bigg[(A_{02}^{(2)}\cos 2ny)c_{10}^{(5/2)}\sin\phi\frac{x}{\sqrt{\varepsilon}}\exp\left(-\alpha\frac{x}{\sqrt{\varepsilon}}\right) + (A_{02}^{(2)}\cos 2ny)c_{10}^{(5/2)} \\ & \times\sin\phi\frac{\pi - x}{\sqrt{\varepsilon}}\exp\left(-\alpha\frac{\pi - x}{\sqrt{\varepsilon}}\right)\Bigg] + \varepsilon^3[C_{11}^{(3)}\cos mx\sin ny] + \varepsilon^4[C_{11}^{(4)}\cos mx\sin ny + C_{20}^{(4)}\sin 2mx \\ & + C_{13}^{(4)}\cos mx\sin 3ny] + O(\varepsilon^5) \end{aligned} \tag{47}$$

$$\Psi_y = \varepsilon^2[D_{11}^{(2)}\sin mx\cos ny] + \varepsilon^3\Big[D_{11}^{(3)}\sin mx\cos ny + D_{02}^{(3)}\sin 2ny - (A_{02}^{(2)}2n\beta\sin 2ny)\Big(d_{01}^{(3)}\cos\phi\frac{x}{\sqrt{\varepsilon}} + d_{10}^{(3)}\sin\phi\frac{x}{\sqrt{\varepsilon}}\Big)\exp\Big(-\alpha\frac{x}{\sqrt{\varepsilon}}\Big) - (A_{02}^{(2)}2n\beta\sin 2ny)\Big(d_{01}^{(3)}\cos\phi\frac{\pi-x}{\sqrt{\varepsilon}} + d_{10}^{(3)}\sin\phi\frac{\pi-x}{\sqrt{\varepsilon}}\Big) \times\exp\Big(-\alpha\frac{\pi-x}{\sqrt{\varepsilon}}\Big)\Big] + \varepsilon^4[D_{11}^{(4)}\sin mx\cos ny + D_{02}^{(4)}\sin 2ny + D_{13}^{(4)}\sin mx\cos 3ny] + O(\varepsilon^5) \tag{48}$$

Next, substituting Eqs. (45)–(48) into the boundary condition (44) and into Eq. (28b), the postbuckling equilibrium paths can be written as

$$\lambda_p = \frac{1}{1+ab_2}\Big[\lambda_p^{(0)} - \lambda_p^{(2)}(A_{11}^{(2)}\varepsilon)^2 + \lambda_p^{(4)}(A_{11}^{(2)}\varepsilon)^4 + \cdots\Big] \tag{49}$$

and

$$\delta_p = \delta_p^{(0)} - \delta_p^{(H)} + \delta_p^{(2)}(A_{11}^{(2)}\varepsilon)^2 + \delta_p^{(4)}(A_{11}^{(2)}\varepsilon)^4 + \cdots \tag{50}$$

In Eqs. (49) and (50), similarly, $(A_{11}^{(2)}\varepsilon)$ is taken as the second perturbation parameter, and we have

$$A_{11}^{(2)}\varepsilon = W_{\rm m} - \Theta_3 W_{\rm m}^2 + \cdots \tag{51a}$$

and the dimensionless maximum deflection of the shell is written as

$$W_{\rm m} = \frac{1}{C_3}\left[\frac{t}{[D_{11}^*D_{22}^*A_{11}^*A_{22}^*]^{1/4}}\frac{\bar{W}}{t} + \Theta_4\right] \tag{51b}$$

It is noted that now $\lambda_q^{(i)}$, $\delta_q^{(i)}$, $\lambda_p^{(i)}$ and $\delta_p^{(i)}$ $(i = 0, 2, 4, \ldots)$ are all functions of temperature and moisture.

Eqs. (40)–(42b) and (49)–(51b) are employed to obtain numerical results for the postbuckling load-shortening or load-deflection curves of moderately thick laminated cylindrical shells subjected to combined loading of axial compression and external pressure under environmental conditions. Buckling under external pressure alone and buckling under axial compression alone follow as two limiting cases. By increasing b_1 and b_2, respectively, the interaction curve of a laminated cylindrical shell under combined loading can be constructed with these two lines. Note that since $b_2 = 1/b_1$, only one load-proportional parameter should be determined in advance. The initial buckling load of a perfect shell can readily be obtained numerically, by setting $\bar{W}^*/t = 0$ (or $\mu = 0$), while taking $\bar{W}/t = 0$ (note that $W_{\rm m} \neq 0$). In this case, the minimum buckling load is determined by considering Eq. (40) or Eq. (49) for various values of the buckling mode (m, n), which determine the number of half-waves in the X-direction and of full waves in the Y-direction. Note that because of Eqs. (36) and (45), the prebuckling deformation of the shell is nonlinear.

4. Numerical results and comments

The efficiency and accuracy of the present method for the buckling and postbuckling analysis of composite laminated cylindrical shells, excluding moisture and temperature effects, were examined by many comparison studies given in Shen (1997a,b,c, 1998). To study the effects of temperature and moisture on the postbuckling behaviour of shear deformable laminated cylindrical shells, several numerical examples were solved for perfect and imperfect, moderately thick, cross-ply laminated cylindrical shells. Graphite/epoxy composite material was selected for the shells in the present examples. However, the analysis is equally applicable to other types of composite material. For these examples $R/t = 30$ and $\bar{Z} = 375$, all plies are of equal thickness and the material properties adopted are (Adams and Miller, 1977; Bowles and Tompkins, 1989): $E_f = 230.0$ GPa, $G_f = 9.0$ GPa, $\nu_f = 0.203$, $\alpha_f = -0.54 \times 10^{-6}$/°C, $\rho_f = 1750$ kg/m^3, $c_{fm} = 0$, $\nu_m = 0.34$, $\alpha_m = 45.0 \times 10^{-6}$/°C, $\rho_m = 1200$ kg/m^3, $\beta_m = 2.68 \times 10^{-3}$/wt.% H_2O and $E_m = (3.51 - 0.003T - 0.142C)$ GPa, in which $T = T_0 + \Delta T$ and $T_0 = 25$°C (room temperature), and $C = C_0 + \Delta C$ and $C_0 = 0$ wt.% H_2O. The numerical results are presented both in tabular and in dimensionless graphical forms. It should be noted that in all figures $\bar{W}^*/t$ and $\bar{W}/t$ mean the dimensionless forms of, respectively, the maximum initial geometric imperfection and additional deflection of the shell.

The buckling loads (σ_{cr}, q_{cr}) (N/mm^2) for perfect 4-ply (0/90/90/0) (or $(0/90)_S$) symmetric cross-ply and (0/90/0/90) (or $(0/90)_{2T}$) antisymmetric cross-ply laminated cylindrical shells under four sets of combined loading conditions, i.e. lateral pressure alone ($b_1 = 0$), combined loading case (1) ($b_1 = 10$), combined loading case (2) ($b_2 = 0.02$) and axial compression alone ($b_2 = 0$), and under three sets of environmental conditions, referred to as 1, 2 and 3, are calculated and compared in Table 1. For environmental case 1, $T = 25$°C, so that both ΔT and ΔC are zero. For environmental case 2, $\Delta T = 50$°C and $\Delta C = 0.5$%, and for environmental case 3, $\Delta T = 100$°C and $\Delta C = 1$%. Also, three values of the fibre volume fraction $V_f (= 0.5$, 0.6 and 0.7) are considered. It is seen that the buckling loads are reduced with increases in moisture and temperature and with decreases in fibre volume fraction.

Fig. 1 shows the interaction between λ_p^* and λ_q^* for $(0/90)_S$ and $(0/90)_{2T}$ laminated cylindrical shells under three environmental conditions, in which $\lambda_q^* = q/q_{cr}$ and $\lambda_p^* = \sigma_x/\sigma_{cr}$, where q_{cr} and σ_{cr} are critical buckling

Table 1
Comparisons of buckling loads (σ_{cr}, q_{cr}) (N/mm^2) for perfect $(0/90)_S$ and $(0/90)_{2T}$ laminated cylindrical shells under combined loading of axial compression and lateral pressure, and under three sets of environmental conditions ($R/t = 30$ and $\bar{Z} = 375$)

Lay-up		$\Delta T = 0$°C, $\Delta C = 0$%	$\Delta T = 50$°C, $\Delta C = 0.5$%	$\Delta T = 100$°C, $\Delta C = 1$%
$(0/90)_S$	$V_f = 0.5$	(279.585, 0)	(272.865, 0)	(266.023, 0)
		(255.708, 0.341)	(249.352, 0.332)	(242.869, 0.324)
		(118.945, 0.793)	(116.571, 0.777)	(114.145, 0.761)
		(0, 1.086)	(0, 1.066)	(0, 1.046)
	$V_f = 0.6$	(332.188, 0)	(324.593, 0)	(316.837, 0)
		(304.295, 0.406)	(297.055, 0.396)	(289.654, 0.386)
		(142.437, 0.950)	(139.670, 0.931)	(136.840, 0.912)
		(0, 1.303)	(0, 1.280)	(0, 1.256)
	$V_f = 0.7$	(399.323, 0)	(390.719, 0)	(381.904, 0)
		(367.150, 0.490)	(358.893, 0.479)	(350.416, 0.467)
		(172.019, 1.147)	(168.756, 1.125)	(165.417, 1.103)
		(0, 1.575)	(0, 1.547)	(0, 1.518)
$(0/90)_{2T}$	$V_f = 0.5$	(297.251, 0)	(290.136, 0)	(282.875, 0)
		(284.665, 0.380)	(277.965, 0.371)	(271.124, 0.361)
		(200.637, 1.338)	(197.283, 1.315)	(193.811, 1.292)
		(0, 2.003)	(0, 1.976)	(0, 1.947)
	$V_f = 0.6$	(352.663, 0)	(344.669, 0)	(336.485, 0)
		(337.805, 0.450)	(330.272, 0.440)	(322.557, 0.430)
		(238.934, 1.593)	(235.118, 1.567)	(231.162, 1.541)
		(0, 2.390)	(0, 2.358)	(0, 2.358)
	$V_f = 0.7$	(422.865, 0)	(413.932, 0)	(404.749, 0)
		(404.916, 0.540)	(396.495, 0.529)	(387.835, 0.517)
		(284.511, 1.897)	(280.256, 1.868)	(275.837, 1.839)
		(0, 2.838)	(0, 2.803)	(0, 2.766)

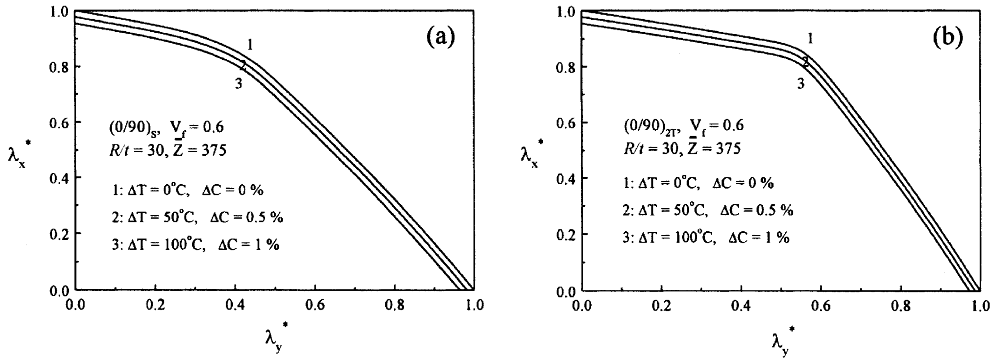

Fig. 1. Hygrothermal effects on the interaction buckling curves for laminated cylindrical shells under combined loading: (a) $(0/90)_S$ and (b) $(0/90)_{2T}$.

loads for the shell with $V_f = 0.6$ under lateral pressure alone or axial compression alone, and under environmental condition $\Delta T = 0°C$, $\Delta C = 0\%$. The results calculated show that for the $(0/90)_S$ shell buckling occurs due to axial compression alone with a buckling mode $(m, n) = (3, 4)$ while due to lateral pressure alone the buckling mode $(m, n) = (1, 3)$. In contrast, for the $(0/90)_{2T}$ shell buckling occurs due to axial compression alone with a buckling mode $(m, n) = (4, 4)$ while due to lateral pressure alone the buckling mode $(m, n) = (1, 3)$. Changes in buckling mode are clearly observed by increasing the load-proportional parameter b_2 (or b_1), i.e. $m = 3$ (or 4) becomes $m = 1$. The interaction curve consists of two lines (by increasing b_2 and b_1, respectively) and the transition from one to another is smooth, so that they seem to be as one line. Then Fig. 2 shows the effects of fibre volume fractions on the interaction buckling curves of $(0/90)_S$ and $(0/90)_{2T}$ laminated cylindrical shells under environmental condition 3. It is seen that the hygrothermal environment or fibre volume fraction has a significant effect on the shape of the interaction buckling curves.

Figs. 3 and 4 give, respectively, the postbuckling load-shortening and load–deflection curves for perfect ($\bar{W}^*/t = 0$) and imperfect ($\bar{W}^*/t = 0.1$), $(0/90)_S$ and $(0/90)_{2T}$ laminated cylindrical shells under combined loading case (2) with the load-proportional parameter $b_2 = 0.0$ (referred to as I) and 0.02 (referred to as II), and under three sets of environmental conditions, in which λ_p^* is defined as in Fig. 1 and $\delta_p^* = \Delta_x/\Delta_{cr}$, where

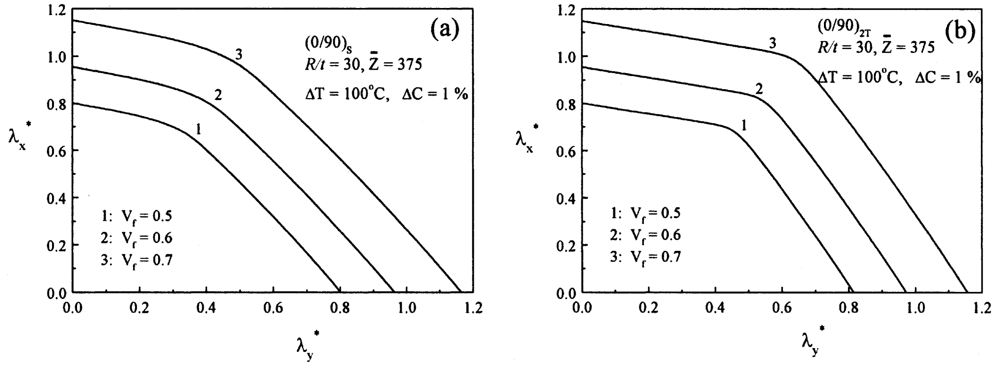

Fig. 2. The effects of fibre volume fractions on the interaction buckling curves for laminated cylindrical shells under combined loading: (a) $(0/90)_S$ and (b) $(0/90)_{2T}$.

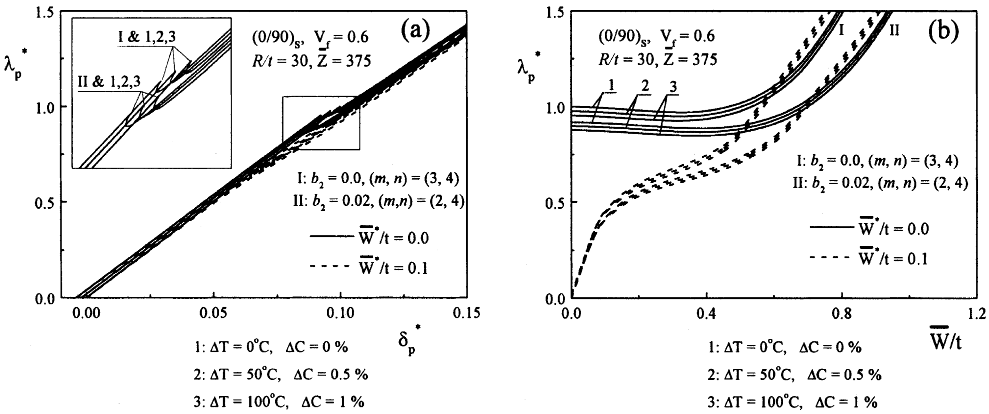

Fig. 3. Hygrothermal effects on the postbuckling of a $(0/90)_S$ laminated cylindrical shell under combined loading: (a) load–shortening and (b) load–deflection.

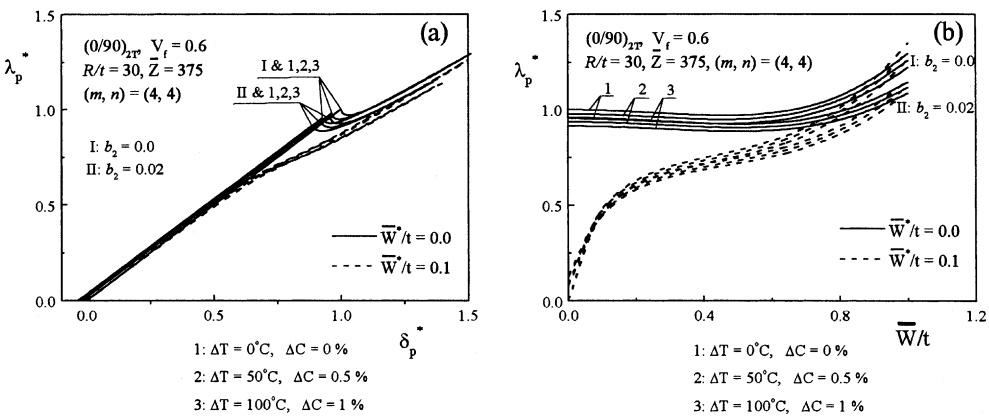

Fig. 4. Hygrothermal effects on the postbuckling of a $(0/90)_{2T}$ laminated cylindrical shell under combined loading: (a) load–shortening and (b) load–deflection.

Δ_{cr} is a critical value of end-shortening corresponding to σ_{cr}. It can be seen that only a weak "snap-through'' phenomenon occurs in the postbuckling range. The elastic limit load can be achieved only for very small imperfections and in such a case imperfection sensitivity can be predicted. In contrast the postbuckling path is stable when $\bar{W}^*/t = 0.1$, and the shell structure becomes imperfection-insensitive. It can also be seen that the buckling loads are reduced with increases in moisture and temperature, and the postbuckling path becomes significantly lower as b_2 increases.

Figs. 5 and 6 show, respectively, the effect of fibre volume fractions $V_f(=0.5$, 0.6 and 0.7) on the postbuckling load-shortening and load–deflection curves for the same two laminated cylindrical shells under combined loading case (2) with the load-proportional parameter $b_2 = 0.0$ and 0.02, and under the environmental condition 3. It can be seen that the buckling loads are reduced with decreases in fibre volume fraction, and the postbuckling path becomes significantly lower as V_f decreases.

The imperfection sensitivity λ^* is calculated and compared in Tables 2 and 3 for $(0/90)_S$ and $(0/90)_{2T}$ laminated cylindrical shells under combined loading case (2) with the load-proportional parameter $b_2 = 0.0$

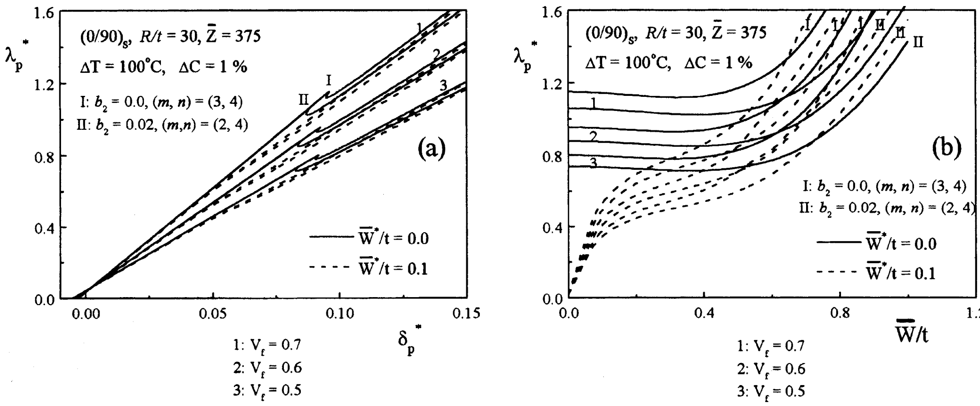

Fig. 5. The effects of fibre volume fractions on the postbuckling of a $(0/90)_S$ laminated cylindrical shell under combined loading: (a) load–shortening and (b) load–deflection.

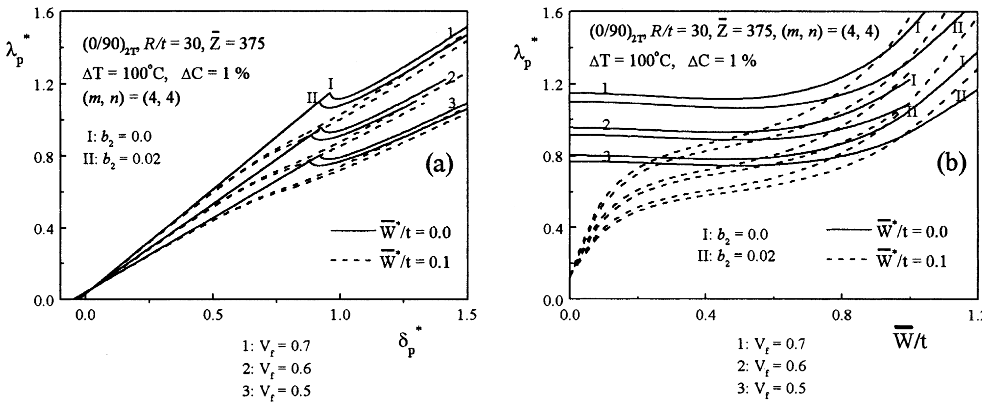

Fig. 6. The effects of fibre volume fractions on the postbuckling of a $(0/90)_{2T}$ laminated cylindrical shell under combined loading: (a) load–shortening and (b) load–deflection.

Table 2
Imperfection sensitivity λ^* for imperfect $(0/90)_S$ laminated cylindrical shells ($V_f = 0.6$, $R/t = 30$ and $\bar{Z} = 375$) under combined loading case (2) and under three sets of environmental conditions

$\bar{W}^*/t$	$\Delta T = 0°C$, $\Delta C = 0\%$		$\Delta T = 50°C$, $\Delta C = 0.5\%$		$\Delta T = 100°C$, $\Delta C = 1\%$	
	$b_2 = 0$	$b_2 = 0.02$	$b_2 = 0$	$b_2 = 0.02$	$b_2 = 0$	$b_2 = 0.02$
0	1.0	1.0	1.0	1.0	1.0	1.0
0.002	0.9740	0.9779	0.9471	0.9780	0.9742	0.9780
0.004	0.9594	0.9653	0.9596	0.9653	0.9598	0.9654
0.006	0.9479	0.9548	0.9473	0.9549	0.9485	0.9550
0.008	–	0.9457	–	0.9457	–	0.9459
0.01	–	0.9374	–	0.9375	–	0.9376
0.012	–	0.9297	–	0.9299	–	0.9300
0.014	–	0.9227	–	0.9228	–	0.9230
0.016	–	0.9160	–	0.9162	–	0.9165
0.018	–	0.9099	–	0.9101	–	0.9104
0.02	–	0.9041	–	0.9044	–	–

Table 3
Imperfection sensitivity λ^* for imperfect $(0/90)_{2T}$ laminated cylindrical shells ($V_f = 0.6$, $R/t = 30$ and $\bar{Z} = 375$) under combined loading case (2) and under three sets of environmental conditions

$\bar{W}^*/t$	$\Delta T = 0°C$, $\Delta C = 0\%$		$\Delta T = 50°C$, $\Delta C = 0.5\%$		$\Delta T = 100°C$, $\Delta C = 1\%$	
	$b_2 = 0$	$b_2 = 0.02$	$b_2 = 0$	$b_2 = 0.02$	$b_2 = 0$	$b_2 = 0.02$
0	1.0	1.0	1.0	1.0	1.0	1.0
0.002	0.9795	0.9798	0.9798	0.9801	0.9802	0.9805
0.004	0.9679	0.9684	0.9684	0.9689	0.9690	0.9695
0.006	0.9586	0.9590	0.9593	0.9597	0.9601	0.9605
0.008	0.9506	0.9510	0.9515	0.9519	0.9525	0.9528
0.01	0.9437	0.9439	–	0.9449	–	0.9461
0.012	–	0.9376	–	–	–	–

and 0.02, and under three sets of environmental conditions. Here, λ^* is the maximum value of σ_x for the imperfect shell, made dimensionless by dividing by the critical value of σ_x for the perfect shell as shown in Table 1. It can be observed that geometric imperfection has a small effect on the buckling loads. These results also show that the imperfection sensitivity of the shell becomes slightly weaker as b_2 and the moisture and temperature increase. Note that the results presented here are only for very small initial geometric imperfections, e.g. $\bar{W}^*/t \leqslant 0.01$ under loading condition of axial compression ($b_2 = 0$), or $\bar{W}^*/t \leqslant 0.02$ under combined loading case (2) ($b_2 = 0.02$).

5. Concluding remarks

In order to assess the effects of temperature and moisture on the postbuckling behaviour of shear deformable laminated cylindrical shell subjected to combined loading of axial compression and external pressure, a fully nonlinear postbuckling analysis is developed based on a micro-macro-mechanical model. The material properties are considered to be dependent on temperature and moisture, which are given explicitly in terms of the fibre and matrix properties and the fibre volume ratio. A boundary layer theory of shell buckling is extended to the case of shear deformable laminated cylindrical shells under hygrothermal environments, and a singular perturbation technique is employed to determine buckling loads and postbuckling equilibrium paths. The numerical examples presented relate to the performance of perfect and imperfect, moderately thick, cross-ply laminated cylindrical shells under different sets of environmental conditions. The results presented herein show that the buckling load and postbuckling strength will degrade and the imperfection sensitivity will become slightly weaker with increasing moisture concentrations and temperatures under the combined loading case (2). In contrast, the imperfection sensitivity can only be predicted by a very small imperfection, and when $\bar{W}^*/t > 0.02$ no elastic limit loads could be found and the shell structure is virtually imperfection insensitive. These results can also provide insight into how the shell parameters and loading conditions interact to affect the buckling load and the postbuckling response.

Acknowledgements

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Appendix A

In Eqs. (1)–(4)

$$
\begin{aligned}
&\tilde{L}_{11}(\ ) = \frac{4}{3t^2}\left[F_{11}^*\frac{\partial^4}{\partial X^4} + (F_{12}^* + F_{21}^* + 4F_{66}^*)\frac{\partial^4}{\partial X^2\,\partial Y^2} + F_{22}^*\frac{\partial^4}{\partial Y^4}\right]\\
&\tilde{L}_{12}(\ ) = \left[D_{11}^* - \frac{4}{3t^2}F_{11}^*\right]\frac{\partial^3}{\partial X^3} + \left[(D_{12}^* + 2D_{66}^*) - \frac{4}{3t^2}(F_{12}^* + 2F_{66}^*)\right]\frac{\partial^3}{\partial X\,\partial Y^2}\\
&\tilde{L}_{13}(\ ) = \left[(D_{12}^* + 2D_{66}^*) - \frac{4}{3t^2}(F_{21}^* + 2F_{66}^*)\right]\frac{\partial^3}{\partial X^2\,\partial Y} + \left[D_{22}^* - \frac{4}{3t^2}F_{22}^*\right]\frac{\partial^3}{\partial Y^3}\\
&\tilde{L}_{14}(\ ) = B_{21}^*\frac{\partial^4}{\partial X^4} + (B_{11}^* + B_{22}^* - 2B_{66}^*)\frac{\partial^4}{\partial X^2\,\partial Y^2} + B_{12}^*\frac{\partial^4}{\partial Y^4}\\
&\tilde{L}_{15}(\bar{N}^{\mathrm{H}}) = \left(B_{11}^*\frac{\partial^2}{\partial X^2} + B_{12}^*\frac{\partial^2}{\partial Y^2}\right)(\bar{N}_x^{\mathrm{H}}) + 2B_{66}^*\frac{\partial^2}{\partial X\,\partial Y}(\bar{N}_{xy}^{\mathrm{H}}) + \left(B_{21}^*\frac{\partial^2}{\partial X^2} + B_{22}^*\frac{\partial^2}{\partial Y^2}\right)(\bar{N}_y^{\mathrm{H}})\\
&\tilde{L}_{16}(\bar{M}^{\mathrm{H}}) = \frac{\partial^2}{\partial X^2}(\bar{M}_x^{\mathrm{H}}) + 2\frac{\partial^2}{\partial X\,\partial Y}(\bar{M}_{xy}^{\mathrm{H}}) + \frac{\partial^2}{\partial Y^2}(\bar{M}_y^{\mathrm{H}})\\
&\tilde{L}_{21}(\ ) = A_{22}^*\frac{\partial^4}{\partial X^4} + (2A_{12}^* + A_{66}^*)\frac{\partial^4}{\partial X^2\,\partial Y^2} + A_{11}^*\frac{\partial^4}{\partial Y^4}\\
&\tilde{L}_{22}(\ ) = \left[B_{21}^* - \frac{4}{3t^2}E_{21}^*\right]\frac{\partial^3}{\partial X^3} + \left[(B_{11}^* - B_{66}^*) - \frac{4}{3t^2}(E_{11}^* - E_{66}^*)\right]\frac{\partial^3}{\partial X\,\partial Y^2}\\
&\tilde{L}_{23}(\ ) = \left[(B_{22}^* - B_{66}^*) - \frac{4}{3t^2}(E_{22}^* - E_{66}^*)\right]\frac{\partial^3}{\partial X^2\,\partial Y} + \left[B_{12}^* - \frac{4}{3t^2}E_{12}^*\right]\frac{\partial^3}{\partial Y^3}\\
&\tilde{L}_{24}(\ ) = \frac{4}{3t^2}\left[E_{21}^*\frac{\partial^4}{\partial X^4} + (E_{11}^* + E_{22}^* - 2E_{66}^*)\frac{\partial^4}{\partial X^2\,\partial Y^2} + E_{12}^*\frac{\partial^4}{\partial Y^4}\right]\\
&\tilde{L}_{25}(\bar{N}^{\mathrm{H}}) = \left(A_{12}^*\frac{\partial^2}{\partial X^2} + A_{11}^*\frac{\partial^2}{\partial Y^2}\right)(\bar{N}_x^{\mathrm{H}}) - A_{66}^*\frac{\partial^2}{\partial X\,\partial Y}(\bar{N}_{xy}^{\mathrm{H}}) + \left(A_{22}^*\frac{\partial^2}{\partial X^2} + A_{12}^*\frac{\partial^2}{\partial Y^2}\right)(\bar{N}_y^{\mathrm{H}})\\
&\tilde{L}_{31}(\ ) = \left[A_{55} - \frac{8}{t^2}D_{55} + \frac{16}{t^4}F_{55}\right]\frac{\partial}{\partial X} + \frac{4}{3t^2}\left[\left(F_{11}^* - \frac{4}{3t^2}H_{11}^*\right)\frac{\partial^3}{\partial X^3} + \left((F_{21}^* + 2F_{66}^*) - \frac{4}{3t^2}(H_{12}^* + 2H_{66}^*)\right)\frac{\partial^3}{\partial X\,\partial Y^2}\right]\\
&\tilde{L}_{32}(\ ) = \left[A_{55} - \frac{8}{t^2}D_{55} + \frac{16}{t^4}F_{55}\right] - \left[D_{11}^* - \frac{8}{3t^2}F_{11}^* + \frac{16}{9t^4}H_{11}^*\right]\frac{\partial^2}{\partial X^2} - \left[D_{66}^* - \frac{8}{3t^2}F_{66}^* + \frac{16}{9t^4}H_{66}^*\right]\frac{\partial^2}{\partial Y^2}\\
&\tilde{L}_{33}(\ ) = \left[(D_{12}^* + D_{66}^*) - \frac{4}{3t^2}(F_{12}^* + F_{21}^* + 2F_{66}^*) + \frac{16}{9t^4}(H_{12}^* + H_{66}^*)\right]\frac{\partial^2}{\partial X\,\partial Y}\\
&\tilde{L}_{34}(\ ) = \tilde{L}_{22}(\ )\\
&\tilde{L}_{35}(\bar{N}^{\mathrm{H}}) = \frac{\partial}{\partial X}\left[\left(B_{11}^* - \frac{4}{3t^2}E_{11}^*\right)\bar{N}_x^{\mathrm{H}} + \left(B_{21}^* - \frac{4}{3t^2}E_{21}^*\right)\bar{N}_y^{\mathrm{H}}\right] + \frac{\partial}{\partial Y}\left[\left(B_{66}^* - \frac{4}{3t^2}E_{66}^*\right)\bar{N}_{xy}^{\mathrm{H}}\right]\\
&\tilde{L}_{36}(\bar{S}^{\mathrm{H}}) = \frac{\partial}{\partial X}(\bar{S}_x^{\mathrm{H}}) + \frac{\partial}{\partial Y}(\bar{S}_{xy}^{\mathrm{H}})\\
&\tilde{L}_{41}(\ ) = \left[A_{44} - \frac{8}{t^2}D_{44} + \frac{16}{t^4}F_{44}\right]\frac{\partial}{\partial Y} + \frac{4}{3t^2}\left[\left((F_{12}^* + 2F_{66}^*) - \frac{4}{3t^2}(H_{12}^* + 2H_{66}^*)\right)\frac{\partial^3}{\partial X^2\,\partial Y} + \left(F_{22}^* - \frac{4}{3t^2}H_{22}^*\right)\frac{\partial^3}{\partial Y^3}\right]\\
&\tilde{L}_{42}(\ ) = \tilde{L}_{33}(\ )\\
&\tilde{L}_{43}(\ ) = \left[A_{44} - \frac{8}{t^2}D_{44} + \frac{16}{t^4}F_{44}\right] - \left[D_{66}^* - \frac{8}{3t^2}F_{66}^* + \frac{16}{9t^4}H_{66}^*\right]\frac{\partial^2}{\partial X^2} - \left[D_{22}^* - \frac{8}{3t^2}F_{22}^* + \frac{16}{9t^4}H_{22}^*\right]\frac{\partial^2}{\partial Y^2}\\
&\tilde{L}_{44}(\ ) = \tilde{L}_{23}(\ )\\
&\tilde{L}_{45}(\bar{N}^{\mathrm{H}}) = \frac{\partial}{\partial X}\left[\left(B_{66}^* - \frac{4}{3t^2}E_{66}^*\right)\bar{N}_{xy}^{\mathrm{H}}\right] + \frac{\partial}{\partial Y}\left[\left(B_{12}^* - \frac{4}{3t^2}E_{12}^*\right)\bar{N}_x^{\mathrm{H}} + \left(B_{22}^* - \frac{4}{3t^2}E_{22}^*\right)\bar{N}_y^{\mathrm{H}}\right]\\
&\tilde{L}_{46}(\bar{S}^{\mathrm{H}}) = \frac{\partial}{\partial X}(\bar{S}_{xy}^{\mathrm{H}}) + \frac{\partial}{\partial Y}(\bar{S}_y^{\mathrm{H}})\\
&\tilde{L}(\ ) = \frac{\partial^2}{\partial X^2}\frac{\partial^2}{\partial Y^2} - 2\frac{\partial^2}{\partial X\,\partial Y}\frac{\partial^2}{\partial X\,\partial Y} + \frac{\partial^2}{\partial Y^2}\frac{\partial^2}{\partial X^2}
\end{aligned}
\tag{A.1}
$$

Appendix B

In Eqs. (25), (27) and (28)

$$
\begin{aligned}
&(\gamma_{110},\gamma_{112},\gamma_{114}) = (4/3t^2)[F^*_{11},(F^*_{12}+F^*_{21}+4F^*_{66})/2,F^*_{22}]/D^*_{11},\\
&(\gamma_{120},\gamma_{122}) = [D^*_{11}-4F^*_{11}/3t^2,(D^*_{12}+2D^*_{66})-4(F^*_{12}+2F^*_{66})/3t^2]/D^*_{11}\\
&(\gamma_{131},\gamma_{133}) = [(D^*_{12}+2D^*_{66})-4(F^*_{21}+2F^*_{66})/3t^2,D^*_{22}-4F^*_{22}/3t^2]/D^*_{11}\\
&(\gamma_{140},\gamma_{142},\gamma_{144}) = [B^*_{21},(B^*_{11}+B^*_{22}-2B^*_{66})/2,B^*_{12}]/[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/4}\\
&(\gamma_{212},\gamma_{214}) = (A^*_{12}+A^*_{66}/2,A^*_{11})/A^*_{22}\\
&(\gamma_{220},\gamma_{222}) = [B^*_{21}-4E^*_{21}/3t^2,(B^*_{11}-B^*_{66})-4(E^*_{11}-E^*_{66})/3t^2]/[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/4}\\
&(\gamma_{231},\gamma_{233}) = [(B^*_{22}-B^*_{66})-4(E^*_{22}-E^*_{66})/3t^2,B^*_{12}-4E^*_{12}/3t^2]/[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/4}\\
&(\gamma_{240},\gamma_{242},\gamma_{244}) = [E^*_{21},(E^*_{11}+E^*_{22}-2E^*_{66})/2,E^*_{12}]/[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/4}\\
&(\gamma_{310},\gamma_{312}) = (4/3t^2)[F^*_{11}-4H^*_{11}/3t^2,(F^*_{21}+2F^*_{66})-4(H^*_{12}+2H^*_{66})/3t^2]/D^*_{11}\\
&(\gamma_{320},\gamma_{322}) = (D^*_{11}-8F^*_{11}/3t^2+16H^*_{11}/9t^4,D^*_{66}-8F^*_{66}/3t^2+16H^*_{66}/9t^4)/D^*_{11}\\
&\gamma_{331} = [(D^*_{12}+D^*_{66})-4(F^*_{12}+F^*_{21}+2F^*_{66})/3t^2+16(H^*_{12}+H^*_{66})/9t^4]/D^*_{11}\\
&(\gamma_{411},\gamma_{413}) = (4/3t^2)[(F^*_{12}+2F^*_{66})-4(H^*_{12}+2H^*_{66})/3t^2,F^*_{22}-4H^*_{22}/3t^2]/D^*_{11}\\
&(\gamma_{430},\gamma_{432}) = (D^*_{66}-8F^*_{66}/3t^2+16H^*_{66}/9t^4,D^*_{22}-8F^*_{22}/3t^2+16H^*_{22}/9t^4)/D^*_{11}\\
&(\gamma_{511},\gamma_{522}) = (B^*_{11}-4E^*_{11}/3t^2,B^*_{22}-4E^*_{22}/3t^2)/[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/4}\\
&(\gamma_{611},\gamma_{622}) = (4/3t^2)(E^*_{11},E^*_{22})/[D^*_{11}D^*_{22}A^*_{11}A^*_{22}]^{1/4}
\end{aligned}
\tag{B.1}
$$

Appendix C

In Eqs. (40)–(42b)

$$\Theta_1 = \frac{1}{C_3\gamma_{24}}\left(1-\frac{1}{2}a_1\gamma_5\right)\lambda_q^{(2)}$$

$$\Theta_2 = \frac{1}{\gamma_{24}}[(\gamma_{T2}-\gamma_5\gamma_{T1})\Delta T + (\gamma_{m2}-\gamma_5\gamma_{m1})\Delta C]\varepsilon - \frac{1}{\gamma_{24}}\left(1-\frac{1}{2}a_1\gamma_5\right)\lambda_q^{(0)}$$

$$\lambda_q^{(0)} = \frac{1}{4}(3)^{3/4}\varepsilon^{-3/2}\left\{\frac{\gamma_{24}m^4}{C_1(1+\mu)g_{06}} + \frac{\gamma_{24}m^2}{C_1(1+\mu)^2}\frac{g_{05}+(1+\mu)g_{07}}{g_{06}}\varepsilon \right.$$
$$+\frac{1}{\gamma_{14}C_1(1+\mu)}\left[g_{08}+\gamma_{14}\gamma_{24}\frac{g_{05}}{g_{06}}\frac{(1+\mu)g_{07}-\mu(2+\mu)g_{05}}{(1+\mu)^2}\right]\varepsilon^2 - \frac{\mu}{(1+\mu)^2}\frac{g_{05}}{\gamma_{14}m^2C_1}\left[1+\frac{g_{05}}{(1+\mu)m^2}\varepsilon\right]$$
$$\left.\times\left[g_{08}+\gamma_{14}\gamma_{24}\frac{g_{05}}{g_{06}}\frac{(1+\mu)g_{07}+g_{05}}{(1+\mu)^2}(2+\mu)\right]\varepsilon^3\right\}$$

$$\lambda_q^{(2)} = \frac{1}{16}(3)^{3/4}\varepsilon^{-3/2}\frac{m^4n^2\beta^2}{g_{06}}\left\{2\gamma_{24}(1+\mu)(2+\mu)+\frac{1}{4}\frac{\gamma_{24}g_{06}g_{13}}{n^2\beta^2C_1}(1+2\mu) - \frac{\gamma_{24}n^2\beta^2g_{06}}{C_1(1+\mu)g_{06}-2a_1m^6g_{10}}\right.$$
$$\left.\times\left[2(1+\mu)^2+\frac{1}{2}\frac{a_1m^2}{C_1}(1+2\mu)+\frac{(1+2\mu)g_{06}+8m^4g_{10}(1+\mu)}{g_{06}}(2+\mu)\right]\right\}$$

$$\delta_q^{(0)} = \frac{1}{\gamma_{24}}\left[\left(\frac{1}{2}a_1\gamma_{24}^2-\gamma_5\right)+\frac{2}{\pi}\frac{\gamma_5}{\gamma_{24}}\left(1-\frac{1}{2}a_1\gamma_5\right)\left(\alpha b_{01}^{(5/2)}-\phi b_{10}^{(5/2)}\right)\varepsilon^{1/2}\right]\lambda_q + \left[\frac{1}{\pi(3)^{3/4}\gamma_{24}^2}\frac{b_{11}}{\alpha}\left(1-\frac{1}{2}a_1\gamma_5\right)^2\varepsilon\right]\lambda_q^2$$

$$\delta_q^{(2)} = \frac{1}{32}(3)^{3/4}\left[m^2(1+2\mu)\varepsilon^{-3/2}-2g_{05}\varepsilon^{-1/2}+\frac{g_{05}^2}{m^2}\varepsilon^{1/2}\right]$$

$$\delta_q^{(H)} = \frac{1}{4\gamma_{24}}(3)^{3/4}\varepsilon^{-1/2}\left[(\gamma_{24}^2\gamma_{T1}-\gamma_5\gamma_{T2})\Delta T + (\gamma_{24}^2\gamma_{\mathrm{m1}}-\gamma_5\gamma_{\mathrm{m2}})\Delta C\right]$$

$$C_1 = n^2\beta^2+\frac{1}{2}a_1m^2,\qquad C_3 = 1-\frac{g_{05}}{m^2}\varepsilon \tag{C.1}$$

and in Eqs. (49)–(51b)

$$\Theta_3 = \frac{1}{C_3}\left[\gamma_{14}\gamma_{24}\frac{m^4(1+\mu)}{16n^2\beta^2g_{09}g_{06}}\varepsilon^{-1}-\gamma_{14}\gamma_{24}\frac{m^2g_{11}}{32n^2\beta^2g_{09}}+\frac{2(\gamma_5-a_2)}{\gamma_{24}}\lambda_p^{(2)}\right]$$
$$\Theta_4 = \frac{1}{\gamma_{24}}[(\gamma_{T2}-\gamma_5\gamma_{T1})\Delta T+(\gamma_{m2}-\gamma_5\gamma_{m1})\Delta C]+\frac{2(\gamma_5-a_2)}{\gamma_{24}}\lambda_p^{(0)} \tag{C.2}$$

$$\lambda_p^{(0)} = \frac{1}{2}C_2\left\{\frac{\gamma_{24}m^2}{(1+\mu)g_{06}}\varepsilon^{-1}+\gamma_{24}\frac{g_{05}+(1+\mu)g_{07}}{(1+\mu)^2g_{06}}+\frac{1}{\gamma_{14}(1+\mu)m^2}\right.$$
$$\times\left[g_{08}+\gamma_{14}\gamma_{24}\frac{g_{05}}{g_{06}}\frac{(1+\mu)g_{07}-\mu(2+\mu)g_{05}}{(1+\mu)^2}\right]\varepsilon-\frac{\mu}{(1+\mu)^2}\frac{g_{05}}{\gamma_{14}m^4}\left[1+\frac{g_{05}}{(1+\mu)m^2}\varepsilon\right]$$
$$\left.\times\left[g_{08}+\gamma_{14}\gamma_{24}\frac{g_{05}}{g_{06}}\frac{g_{05}+(1+\mu)g_{07}}{(1+\mu)^2}(2+\mu)\right]\varepsilon^2\right\}$$

$$\lambda_p^{(2)} = \frac{1}{8}C_2\left\{\gamma_{14}\gamma_{24}^2\frac{m^6(2+\mu)}{2g_{09}g_{06}^2}\varepsilon^{-1} + \gamma_{14}\gamma_{24}\frac{m^4}{2g_{09}g_{06}}\left[\frac{g_{05}}{g_{06}}\frac{1}{1+\mu} + \frac{g_{07}}{g_{06}}(1+\mu) + g_{12}(1+\mu) - \frac{1}{2}\frac{(2+\mu)}{(1+\mu)}g_{11}\right]\right.$$
$$- \frac{1}{4}\gamma_{24}m^2g_{13}(1+2\mu)\varepsilon + \gamma_{14}\gamma_{24}^2\frac{m^2g_{11}}{2g_{09}}\left[\frac{g_{05}}{g_{06}}\frac{1}{1+\mu} - \frac{g_{07}}{g_{06}} - g_{12}\right]\varepsilon + \gamma_{14}\gamma_{24}^2\frac{m^2g_{05}}{2g_{09}g_{06}}$$
$$\left.\times\left[\frac{2(1+\mu)^2-(1+2\mu)}{2(1+\mu)^2}g_{14} + \frac{\mu}{1+\mu}\frac{g_{05}}{g_{06}}\right](2+\mu)\varepsilon + \gamma_{24}\frac{m^2n^4\beta^4}{g_{06}}\frac{S_2}{S_1}\varepsilon\right\}$$

$$\lambda_p^{(4)} = \frac{1}{128}C_2\gamma_{14}^2\gamma_{24}^3\frac{m^{10}(1+\mu)}{g_{09}^2g_{06}^3}\frac{S_3}{S_{13}}\varepsilon^{-1}$$

$$\delta_p^{(0)} = \frac{(1+ab_2)}{\gamma_{24}}\left[(\gamma_{24}^2 - a_2\gamma_5) - \frac{2}{\pi}\frac{\gamma_5(\gamma_5-a_2)}{\gamma_{24}}(\alpha b_{01}^{(2)} - \phi b_{10}^{(2)})\varepsilon^{1/2}\right]\lambda_p + \left[\frac{b_{11}}{2\pi\alpha}\frac{(a_2-\gamma_5)^2}{\gamma_{24}^2}(1+ab_2)^2\varepsilon^{1/2}\right]\lambda_p^2$$

$$\delta_p^{(2)} = \frac{1}{16}\left[m^2(1+2\mu)\varepsilon - 2g_{05}\varepsilon^2 + \frac{g_{05}^2}{m^2}\varepsilon^3\right]$$

$$\delta_p^{(4)} = \frac{1}{128}\left\{\frac{b_{11}}{32\pi\alpha}\gamma_{14}^2\gamma_{24}^2\frac{m^8(1+\mu)^2}{n^4\beta^4g_{09}^2g_{06}^2}\varepsilon^{-3/2} + m^2n^4\beta^4(1+\mu)^2\left(\frac{S_4}{S_1}\right)^2\varepsilon^3\right\}$$

$$\delta_p^{(H)} = \frac{1}{2\gamma_{24}}\left[(\gamma_{24}^2\gamma_{T1} - \gamma_5\gamma_{T2})\Delta T + (\gamma_{24}^2\gamma_{m1} - \gamma_5\gamma_{m2})\Delta C\right]$$

$$S_1 = g_{06}(1+\mu) - 4m^2C_2g_{10}$$

$$S_2 = g_{06}[(4+9\mu+4\mu^2) + C_2(1+2\mu)] + 8m^4(1+\mu)(2+\mu)g_{10}$$

$$S_3 = g_{136}[C_9(1+3\mu+\mu^2) + C_5(4+2\mu) + (1+\mu)] + g_{06}[C_5(6+8\mu+2\mu^2) - (2\mu+3\mu^2+\mu^3)]$$

$$S_4 = g_{06}(1+2\mu) + 8m^4(1+\mu)g_{10}$$

$$S_{13} = g_{136}C_9 - g_{06}(1+\mu)$$

$$C_2 = \frac{m^2}{m^2+a_2n^2\beta^2}, \qquad C_5 = \frac{m^2+5a_2n^2\beta^2}{m^2+a_2n^2\beta^2}, \qquad C_9 = \frac{m^2+9a_2n^2\beta^2}{m^2+a_2n^2\beta^2}$$

in the above equations

$$g_{00} = (\gamma_{31} + \gamma_{320}m^2 + \gamma_{322}n^2\beta^2)(\gamma_{41} + \gamma_{430}m^2 + \gamma_{432}n^2\beta^2) - \gamma_{331}^2 m^2 n^2 \beta^2$$

$$g_{01} = (\gamma_{41} + \gamma_{430}m^2 + \gamma_{432}n^2\beta^2)(\gamma_{220}m^2 + \gamma_{222}n^2\beta^2) - \gamma_{331}n^2\beta^2(\gamma_{231}m^2 + \gamma_{233}n^2\beta^2)$$

$$g_{02} = (\gamma_{31} + \gamma_{320}m^2 + \gamma_{322}n^2\beta^2)(\gamma_{231}m^2 + \gamma_{233}n^2\beta^2) - \gamma_{331}m^2(\gamma_{220}m^2 + \gamma_{222}n^2\beta^2)$$

$$g_{03} = (\gamma_{31} + \gamma_{320}m^2 + \gamma_{322}n^2\beta^2)(\gamma_{41} - \gamma_{411}m^2 - \gamma_{413}n^2\beta^2) - \gamma_{331}m^2(\gamma_{31} - \gamma_{310}m^2 - \gamma_{312}n^2\beta^2)$$

$$g_{04} = (\gamma_{41} + \gamma_{430}m^2 + \gamma_{432}n^2\beta^2)(\gamma_{31} - \gamma_{310}m^2 - \gamma_{312}n^2\beta^2) - \gamma_{331}n^2\beta^2(\gamma_{41} - \gamma_{411}m^2 - \gamma_{413}n^2\beta^2)$$

$$g_{05} = (\gamma_{240}m^4 + 2\gamma_{242}m^2n^2\beta^2 + \gamma_{244}n^4\beta^4) + \frac{m^2(\gamma_{220}m^2 + \gamma_{222}n^2\beta^2)g_{04} + n^2\beta^2(\gamma_{231}m^2 + \gamma_{233}n^2\beta^2)g_{03}}{g_{00}}$$

$$g_{06} = (m^4 + 2\gamma_{212}m^2n^2\beta^2 + \gamma_{214}n^4\beta^4) + \gamma_{14}\gamma_{24}\frac{m^2(\gamma_{220}m^2 + \gamma_{222}n^2\beta^2)g_{01} + n^2\beta^2(\gamma_{231}m^2 + \gamma_{233}n^2\beta^2)g_{02}}{g_{00}}$$

$$g_{07} = (\gamma_{140}m^4 + 2\gamma_{142}m^2n^2\beta^2 + \gamma_{144}n^4\beta^4) - \frac{m^2(\gamma_{120}m^2 + \gamma_{122}n^2\beta^2)g_{01} + n^2\beta^2(\gamma_{131}m^2 + \gamma_{133}n^2\beta^2)g_{02}}{g_{00}}$$

$$g_{08} = (\gamma_{110}m^4 + 2\gamma_{112}m^2n^2\beta^2 + \gamma_{114}n^4\beta^4) + \frac{m^2(\gamma_{120}m^2 + \gamma_{122}n^2\beta^2)g_{04} + n^2\beta^2(\gamma_{131}m^2 + \gamma_{133}n^2\beta^2)g_{03}}{g_{00}}$$

$$g_{10} = 1 + \gamma_{14}\gamma_{24}\gamma_{220}^2\frac{4m^2}{\gamma_{31} + \gamma_{320}4m^2}$$

$$g_{12} = \frac{\gamma_{244}(\gamma_{41} + \gamma_{432}4n^2\beta^2) + \gamma_{233}(\gamma_{41} - \gamma_{413}4n^2\beta^2)}{\gamma_{214}(\gamma_{41} + \gamma_{432}4n^2\beta^2) + \gamma_{14}\gamma_{24}\gamma_{233}^2 4n^2\beta^2}$$

$$g_{12}^* = \frac{\gamma_{214}(\gamma_{41} - \gamma_{413}4n^2\beta^2) - \gamma_{14}\gamma_{24}\gamma_{233}\gamma_{244}4n^2\beta^2}{\gamma_{214}(\gamma_{41} + \gamma_{432}4n^2\beta^2) + \gamma_{14}\gamma_{24}\gamma_{233}^2 4n^2\beta^2}$$

$$g_{09} = \gamma_{114} + \gamma_{133}g_{12}^* + \gamma_{14}\gamma_{24}\gamma_{144}g_{12}$$

$$g_{13} = \frac{\gamma_{41} + \gamma_{432}4n^2\beta^2}{\gamma_{214}(\gamma_{41} + \gamma_{432}4n^2\beta^2) + \gamma_{14}\gamma_{24}\gamma_{233}^2 4n^2\beta^2}$$

$$g_{14} = -\frac{\gamma_{144}(\gamma_{41} + \gamma_{432}4n^2\beta^2) - \gamma_{133}\gamma_{233}4n^2\beta^2}{\gamma_{214}(\gamma_{41} + \gamma_{432}4n^2\beta^2) + \gamma_{14}\gamma_{24}\gamma_{233}^2 4n^2\beta^2}$$

$$g_{11} = g_{14}(1 + 2\mu) + 2\frac{g_{05}}{g_{06}}$$

$$g_{130} = (\gamma_{31} + \gamma_{320}m^2 + \gamma_{322}9n^2\beta^2)(\gamma_{41} + \gamma_{430}m^2 + \gamma_{432}9n^2\beta^2) - \gamma_{331}^2 9m^2n^2\beta^2$$

$$g_{131} = (\gamma_{41} + \gamma_{430}m^2 + \gamma_{432}9n^2\beta^2)(\gamma_{220}m^2 + \gamma_{222}9n^2\beta^2) - \gamma_{331}9n^2\beta^2(\gamma_{231}m^2 + \gamma_{233}9n^2\beta)^2$$

$$g_{132} = (\gamma_{31} + \gamma_{320}m^2 + \gamma_{322}9n^2\beta^2)(\gamma_{231}m^2 + \gamma_{233}9n^2\beta^2) - \gamma_{331}m^2(\gamma_{220}m^2 + \gamma_{222}9n^2\beta^2)$$

$$g_{136} = (m^4 + 18\gamma_{212}m^2n^2\beta^2 + \gamma_{214}81n^4\beta^4) + \gamma_{14}\gamma_{24}\frac{m^2(\gamma_{220}m^2 + \gamma_{222}9n^2\beta^2)g_{131} + 9n^2\beta^2(\gamma_{231}m^2 + \gamma_{233}9n^2\beta^2)g_{132}}{g_{130}}$$

$$g_{15} = \gamma_{220}(\gamma_{310} + \gamma_{120}) - \gamma_{320}(\gamma_{140} + \gamma_{240})$$

$$g_{16} = (\gamma_{320} + \gamma_{14}\gamma_{24}\gamma_{220}^2)(\gamma_{320}\gamma_{110} - \gamma_{310}\gamma_{120}) + \gamma_{14}\gamma_{24}(\gamma_{320}\gamma_{140} - \gamma_{120}\gamma_{220})(\gamma_{320}\gamma_{240} - \gamma_{310}\gamma_{220})$$

$$b = \left[\frac{\gamma_{14}\gamma_{24}\gamma_{320}^2}{g_{16}}\right]^{1/2}, \qquad c = \gamma_{14}\gamma_{24}\gamma_{320}\frac{g_{15}}{2g_{16}}$$

$$\alpha = [(b - c)/2]^{1/2}, \qquad \phi = [(b + c)/2]^{1/2}$$

$$g_{17} = \frac{(\gamma_{310} + \gamma_{14}\gamma_{24}\gamma_{220}\gamma_{240})b - \gamma_{14}\gamma_{24}\gamma_{220}}{(\gamma_{310} + \gamma_{14}\gamma_{24}\gamma_{220}\gamma_{240})b + \gamma_{14}\gamma_{24}\gamma_{220}}$$

$$g_{19} = \frac{\gamma_{320}}{\gamma_{320} + \gamma_{14}\gamma_{24}\gamma_{220}^2}\frac{(2\alpha^2 g_{17} - c)}{b^2} + \frac{\gamma_{310}\gamma_{220} - \gamma_{320}\gamma_{240}}{\gamma_{320} + \gamma_{14}\gamma_{24}\gamma_{220}^2}$$

$$g_{20} = -\frac{\gamma_{320}}{\gamma_{320} + \gamma_{14}\gamma_{24}\gamma_{220}^2}\frac{(2\phi^2 g_{17} + c)}{b^2} + \frac{2}{b}\frac{\gamma_{320}g_{17}}{\gamma_{320} + \gamma_{14}\gamma_{24}\gamma_{220}^2} - \frac{2\gamma_{310}\gamma_{320} - (\gamma_{310}\gamma_{220}g_{17} - \gamma_{320}\gamma_{240})[(\gamma_{310} + \gamma_{14}\gamma_{24}\gamma_{220}\gamma_{240})b + \gamma_{14}\gamma_{24}\gamma_{220}]}{(\gamma_{320} + \gamma_{14}\gamma_{24}\gamma_{220}^2)[(\gamma_{310} + \gamma_{14}\gamma_{24}\gamma_{220}\gamma_{240})b + \gamma_{14}\gamma_{24}\gamma_{220}]}$$

$$a_{01}^{(1)} = a_{01}^{(3/2)} = 1, \qquad a_{10}^{(1)} = a_{10}^{(3/2)} = \frac{\alpha}{\phi}g_{17}$$

$$b_{01}^{(2)} = b_{01}^{(5/2)} = \gamma_{24}g_{19}, \qquad b_{10}^{(2)} = b_{10}^{(5/2)} = \gamma_{24}\frac{\alpha}{\phi}g_{20}$$

$$b_{11} = \frac{1}{b}\left[(a_{10}^{(1)})^2\phi^2 b + a_{10}^{(1)}2\alpha\phi c + (2\alpha^4 - \alpha^2\phi^2 + \phi^4)\right] \tag{C.3}$$

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